

## A Decomposition of $R_p$ into Contributions from single-parameter Reflectivities

Kristopher A. Innanen\*

\*Dept. Geoscience, University of Calgary, CREWES Project, Calgary AB

k.innanen@ucalgary.ca

### Summary

In AVO/AVA inversion, a linearized form of the Zoeppritz equations known as the Aki-Richards approximation and variants are used to model  $R_p$ . This approximation can be viewed as a linear decomposition of the full reflection coefficient into contributions from the reflectivities of individual medium parameters. A forward/inverse series framework leads to an alternative approach to this type of decomposition. The first order terms in the decomposition are qualitatively similar to the Aki-Richards approximation, with second- and third-order terms correcting the approximation at large angle and large contrast. We test the approach both for acoustic and elastic reflection coefficients. In the elastic case, where forward/inverse methods of the kind we use require a consideration of both  $R_p$  and  $R_s$ , we proceed in an approximate fashion using  $R_p$  only. In spite of the approximation, the low order nonlinear terms provide a significant increase in accuracy over the linear/Aki-Richards approximation in several large contrast/large angle model regimes. Separately determining individual reflectivities could provide useful input to bandlimited impedance inversion, or enhance our ability to extrapolate data from small to large angle.

### Introduction

Practical inversion of amplitude information in reflection seismic data (e.g., Castagna, 1993; Downton & Ursenbach, 2006) is based on linear-approximate solutions of the Zoeppritz equations, in particular that of Aki and Richards (2002) (hereafter referred to as AR). Although the Zoeppritz equations can be solved numerically (and even analytically, if you don't mind a mess), the linearized solutions have historically won out over the more complex exact forms as practical tools. One of the reasons for this is that they may be viewed as direct decompositions of the full  $R_p$  coefficient into contributions from reflectivities due to individual parameter variations (e.g., Goodway, 2006). For instance, the AR approximation

$$R_p(\theta) = \frac{1}{2} \left( \frac{\Delta V_p}{V_p} + \frac{\Delta \rho}{\rho} \right) - 2 \frac{V_s^2}{V_p^2} \sin^2 \theta \left( 2 \frac{\Delta V_s}{V_s} + \frac{\Delta \rho}{\rho} \right) + \frac{1}{2} \tan^2 \theta \frac{\Delta V_p}{V_p}, \quad (1)$$

in which  $V_p$ ,  $V_s$  and  $\rho$  represent the mean values of P-wave velocity, S-wave velocity and density respectively across the boundary, can be seen to explicitly express  $R_p$  in terms of

$$\frac{\Delta V_p}{V_p} \approx \frac{V_p^L - V_p^U}{V_p^L + V_p^U}, \quad \frac{\Delta V_s}{V_s} \approx \frac{V_s^L - V_s^U}{V_s^L + V_s^U}, \quad \frac{\Delta \rho}{\rho} \approx \frac{\rho^L - \rho^U}{\rho^L + \rho^U}, \quad (2)$$

where superscripts L and U signify the lower and upper media respectively. These fractions are evidently equivalent to the reflection coefficients at normal incidence that would have been measured had only those individual parameters varied. The power of such a decomposition, beyond the analytical clarity it brings, is that with these reflectivities in hand, well-developed methods for normal-incidence, single-parameter bandlimited impedance inversion (e.g., Walker & Ulrych, 1993) may be straightforwardly employed to

complete the inversion. Still, there is the matter of the inexact nature of AR and the many approximations deriving from it (e.g., Shuey, 1985), and the error that accrues at large contrasts and large angles. There have been several notable attempts to enhance accuracy by for instance providing higher-order corrections to AR. Such corrections have been constructed based on Taylor's series expansions with respect to both the model parameters within the Zoeppritz equations (Ursin, & Dahl, 1992), and with respect to the ray parameter (Wang, 1999). In this paper we take another approach, using the tools of direct inversion, which have been developed of late for the determination of parameter contrasts from reflection coefficients (e.g., Zhang & Weglein, 2009; Innanen, 2011). Here we use them to decompose acoustic and elastic reflection coefficients into their component reflectivities. We discuss a formula for the reconstitution of the full acoustic multiparameter reflection coefficient in terms of several individual reflectivities. Interestingly, within this multiparameter acoustic configuration, the *same formula* is found to approximate  $R_p$ , regardless of which parameters vary, how many of them vary, and regardless of which experimental variable(s)  $R_p$  varies over. The elastic version, quoted next, is at present approximate, but in many regimes of large contrast/angle the accuracy of the formula appears significantly higher than AR and other linearizations.

## Theory

Let  $R_p$  be the reflection coefficient associated with an interface across which  $N$  acoustic parameters,  $\mu = (\mu_1, \mu_2, \dots, \mu_N)$ , have varied, from  $\mu^0$  in the incidence medium, to  $\mu^1$  in the target medium (for instance, these  $\mu$  might represent P-wave velocity and density, in which case  $\mu = (V_p, \rho)$  varies from  $\mu^0 = (V_p^0, \rho^0)$  to  $\mu^1 = (V_p^1, \rho^1)$ ). We introduce  $N$  additional reflection coefficients  $R_{\mu} = (R_{\mu_1}, R_{\mu_2}, \dots, R_{\mu_N})$ , where  $R_{\mu_j}$  is the reflection coefficient associated with an interface across which only  $\mu_j$  has changed (for instance,  $R_p$  is the P-P reflection coefficient associated with an interface across which density varied from  $\rho^0$  to  $\rho^1$ , and all other parameters remained constant).  $R_p$  is expressible, explicitly to third order, as

$$R_p = \sum_{i=1}^N R_{\mu_i} - \frac{1}{3} \left[ \left( \sum_{i=1}^N R_{\mu_i} \right)^3 - \left( \sum_{i=1}^N R_{\mu_i}^3 \right) \right] + \dots, \quad (3)$$

with fifth order and higher corrections straightforwardly available. The series in (3) is an exact expression; the elastic equivalent, in P-, S- and  $\rho$ -reflectivities,  $R_\alpha$ ,  $R_\beta$ , and  $R_\rho$  respectively, is given *approximately* by

$$R_p(\theta) \approx R_1(\theta) + R_2(\theta) + R_3(\theta) + \dots, \quad (4)$$

where

$$\begin{aligned} R_1(\theta) &= R_\alpha + R_\beta + R_\rho, & R_2(\theta) &= W_1 R_\beta^2 + W_2 R_\beta R_\rho, \\ R_3(\theta) &= W_3 R_\alpha^2 R_\beta + W_4 R_\alpha^2 R_\rho + W_5 R_\alpha R_\beta^2 + W_6 R_\beta^2 R_\rho + W_7 R_\beta R_\rho^2 + W_8 R_\alpha R_\rho^2 + W_9 R_\alpha R_\beta R_\rho, \end{aligned} \quad (5)$$

and  $W_1 = (1/BX^2)(1/2 - 1/B) - 1/8$ ,  $W_2 = 1 - 2B$ ,  $W_3 = W_4 = W_8 = -1$ ,  $W_5 = -(1/2)[1/(BX^2)]$ ,  $W_6 = [1/(2X^2)][1 - 9/(2B) + 7/(4B^2)]$ ,  $W_7 = 3/2 - B/2$ , and  $W_9 = 4B - 2$ . Here  $X = \sin \theta$  and  $B = V_S/V_P$ .

The full derivation of equations (3) and (5) is beyond the scope of this abstract; instead let us generate equation (3) for one case; generalization is possible by repeating this for a range of cases and identifying the pattern. We begin with the acoustic  $R$  for a boundary across which density and P-wave velocity vary:

$$R(\theta) = \frac{(\rho_1/\rho_0)(V_p^1/V_p^0)\cos\theta - \sqrt{1 - (V_p^1/V_p^0)^2 \sin^2\theta}}{(\rho_1/\rho_0)(V_p^1/V_p^0)\cos\theta + \sqrt{1 - (V_p^1/V_p^0)^2 \sin^2\theta}}. \quad (6)$$

Defining  $a_p = 1 - (V_p^0/V_p^1)^2$ , and  $a_\rho = 1 - \rho^0/\rho^1$ , and expanding equation (6) in these parameters and  $\sin^2 \theta$ , we have, evaluating at normal incidence,

$$R(\theta=0) = \frac{1}{4}a_p + \frac{1}{2}a_\rho + \frac{1}{8}a_p^2 + \frac{1}{4}a_\rho^2 + \frac{5}{64}a_p^3 + \frac{1}{8}a_\rho^3 - \frac{1}{32}a_p^2a_\rho - \frac{1}{16}a_p a_\rho^2 + \dots \quad (7)$$

We next perform the same expansions on versions of equation (6) in which only one of  $V_P$  or  $\rho$  vary at a time:  $R_{VP} = (V_P^1 - V_P^0)/(V_P^1 + V_P^0)$  and  $R_\rho = (\rho^1 - \rho^0)/(\rho^1 + \rho^0)$ :

$$R_{VP} = \frac{1}{4}a_p + \frac{1}{8}a_p^2 + \frac{5}{64}a_p^3 + \dots, \quad R_\rho = \frac{1}{2}a_\rho + \frac{1}{4}a_\rho^2 + \frac{1}{8}a_\rho^3 + \dots \quad (8)$$

Each of these 1-parameter relationships can be inverted by substituting, into (8), the inverse series  $a_p = a_{p1} + a_{p2} + \dots$ , and  $a_\rho = a_{\rho1} + a_{\rho2} + \dots$ , in which the index  $i$  signifies  $i$ th order in  $R_{VP}$  or  $R_\rho$ , equating like orders, and sequentially solving for  $a_p$  and  $a_\rho$ . The formulas

$$a_p = 4(R_{VP} - 2R_{VP}^2 + 3R_{VP}^3 + \dots), \quad a_\rho = 2(R_\rho - R_\rho^2 + R_\rho^3 + \dots) \quad (9)$$

thus produced, may be used to eliminate the perturbations in equation (7) in favour of  $R_{VP}$  and  $R_\rho$ . Re-introducing variable  $\theta$ , we obtain

$$R(\theta) = R_{VP}(\theta) + R_\rho - R_{VP}^2(\theta)R_\rho - R_{VP}(\theta)R_\rho + \dots \quad (10)$$

which is equivalent to equation (3) evaluated for the 2-parameter, density/velocity case.

### Examples

Let us examine this approach to decomposition in terms of its ability to accurately, and, with relatively low-order truncations, reproduce reflection coefficients at large angle and large contrast. Fig. 1 illustrates the use of equation (10) at various orders. Approximations are plotted for three configurations of medium properties. In each plot, first order (blue), third order (red), and fifth order (green) approximations are compared to the exact reflection coefficient (black). The linearization is often close to the result achieved by the AR approximation (reduced to mimic an acoustic problem), and hence the blue line is a reasonably faithful guide to the accuracy to be expected from AR in each circumstance. By third or fifth order significant improvement is noted.

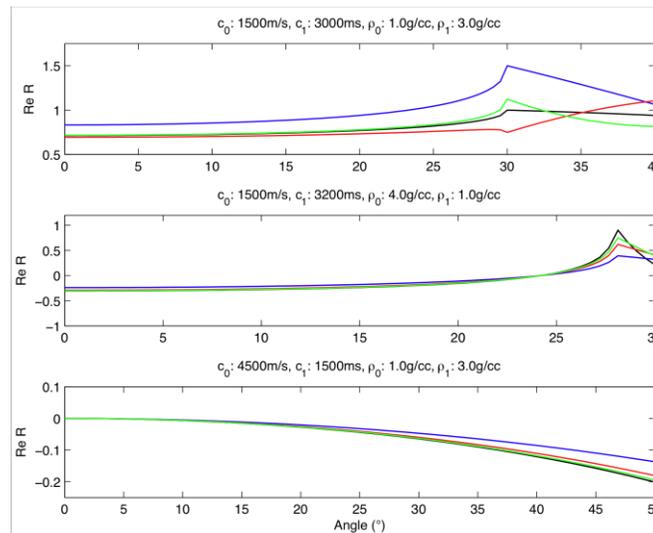


Figure 1: Various truncations of equation (10). Three configurations of medium parameters & large contrast/large angle are shown. Black: exact R; blue: first order; red: third order; green: fifth order.

In Fig. 2 we carry out much the same comparison, but using the elastic approximation in equation (4). Black is exact, blue is linear, and red is third order. Similar increases in accuracy from the linear approximation, which is analogous though not identical to the AR approximation, are noted by third order.

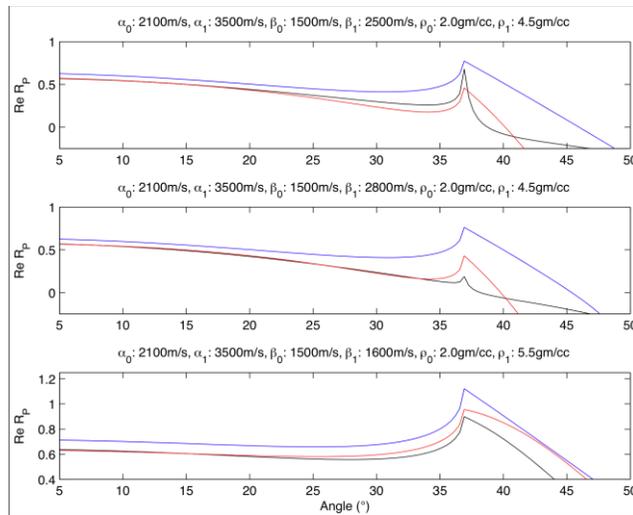


Figure 2: Various truncations of equation (4). Three configurations of medium parameters & large contrast large angle are shown. Black: exact R; blue: first order; red: third order; green: fifth order.

## Conclusions

These approximations describe the fundamentally nonlinear relationships between measurable reflection coefficients and the notional 1-parameter reflectivities which underlie them. There are several ways they could be used. First, via a nonlinear regression the best-fit reflectivities could be estimated at normal incidence. Then, impedance inversion methods could be used on each reflectivity function to determine profiles. Second, the equations could be used to extrapolate data to high angles. Also, there is no reason to limit the reflectivity decompositions to those in  $V_P$ ,  $V_S$  and  $\rho$ . As discussed by Goodway (2006), often Lamé parameters  $\lambda$ ,  $\mu$  and  $\rho$ , or Lamé impedances  $\lambda\rho$  and  $\mu\rho$  are more useful products; this approach would extend readily to include 1-parameter reflectivities with any of these parametrizations. As far as limitations go, what we have developed is an AVA theory; none of the issues of transformation to an AVO theory have been broached as of yet. Also, posing the elastic problem consistently, using both S- and P-reflectivities, thus going beyond equation (4), is a key step currently under investigation.

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