# Gabor Multipliers for Seismic Modeling and Wavefield Propagators

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## Summary

Nonstationary processes in seismic modeling may be represented numerically by Gabor multipliers and generalized frame operators. These numerical algorithms are based on a localized version of the Fourier transform and share many of the speed and accuracy benefits of the FFT, while modeling nonstationarity. We demonstrate applications to seismic deconvolution and numerical wavefield propagation, and describe some to the mathematical results that underlie these methods.

## Introduction

Proper numerical models of seismic processes must account for nonstationary behaviour of signals and systems, reflecting the simple fact that physical properties of rocks and fluids vary within the earth. Gabor multipliers form a mathematical tool that conveniently models nonstationary processes, based on a localized version of the Fourier transform, and allowing the design and analysis of signal processing algorithms in a localized frequency domain. These algorithms can be used as the core of migration or imaging processes.

The guiding principle is to design a suite of stationary operators, each one appropriate for certain portions of the signal (corresponding to an assumed stationary model of the earth on that portion), and combine these operators using a nonstationary decomposition of the signal with localized windowing functions. By a careful choice of local operators and windows, one can develop meaningful numerical methods that accurately model the physics of seismic wave propagation in the earth.

We have developed a mathematical theory for these methods, based on pseudodifferential operator theory, harmonic analysis, and generalized frames. We summarize some of these mathematical results and demonstrate how Gabor multipliers can be used to accurately model seismic Q-attenuation and wave propagation.

## Theory and Method

A key step in numerically localizing a signal is selecting dual sets of windowing functions, the analysis windows $g_k(t)$ and the synthesis windows $h_k(t)$ to span the signal space under study, satisfying the partition of unity condition

$$\sum_{k=1}^{m} \overline{h_k(t)} g_k(t) = 1$$

A generic signal f(t) is decomposed into windowed slices $g_k(t)f(t)$, each one processed separately by a numerical operator $A_k$, and the results recombined through the synthesis windows $h_k$ to create the output

$$Af = \sum_{k=1}^{m} \overline{h_k} A_k(g_k f)$$

The design of operator A is completely determined by the windows and the local operators $A_k$; we use a physical model of the local behaviour of the seismic model to choose windows and local operators. Here, we have described the decomposition of signals with respect to a time variable "t". However, we may instead decompose with respect to space variables, temporal frequency, and/or spatial frequencies. The overall technique is the same.

Mathematically, the properties of such a sum of operators may be described by Stinespring's theorem, and the theory of generalized frames. In particular, we may conclude the following mathematical results:

- The Gabor transform approximately factorizes the action of the Gabor multiplier
- Gabor multipliers accurately model pseudodifferential operators, for slowly varying symbols
- for self-dual windows, a bounded family of operators $A_k$ results in a bounded operator A.

Applied to seismic modeling problems we conclude that

- factorization in the time-frequency domain generalizes Wiener stationary deconvolution;
- seismic Q-attenuation may be quickly and accurately modeled using Gabor multipliers;
- combining stable stationary wavefield propagators leads to accurate, stable nonstationary propagators.

The factorization result may be derived from the following approximation

$$\begin{aligned} \mathcal{G}d(t_k,\omega) &\approx \mathcal{F}([s * a_j * (r \cdot g_k)]) \\ &= \widehat{s}(\omega) \cdot \alpha(t_k,\omega) \cdot \widehat{r \cdot g_k}(\omega) \\ &= \widehat{s}(\omega) \cdot \alpha(t_k,\omega) \cdot \mathcal{G}r(t_k,\omega) \end{aligned}$$

which states that the Gabor transform "Gd(t,w)" of the recorded seismic data "d" is approximately equal to the Fourier transform of the seismic source "s", times the attenuation operator "α" times the Gabor transform "Gr(t,w)". This is the key factorization result that is the basis for Gabor deconvolution. Essentially, the attenuation term "α" may be estimated from the recorded seismic data by assuming a bandlimited source and whiteness in the reflectivity – a decay factor dependent on the product "tω" is the contribution to "α", which can be approximated by smoothing the Gabor transform of the recorded data. Phase information is recovered by a minimum phase assumption, which specifies in detail the form of the Gabor multiplier.

**Examples**

We give two numerical examples to demonstrate the utility of the Gabor method. In the first example, we model seismic Q-attenuation with a pseudodifferential operator whose operator symbol is given in the form

$$\alpha(t,f) = \exp(\pi t f/Q) \cdot \text{phase term}$$

where the phase term is calculated to enforce a minimum phase condition. With this slowly varying symbol, we may approximate using a Gabor multiplier, using windows selected to be relatively small compared to the rate at which the symbol changes. Figure 1 shows the result of a sample implementation.

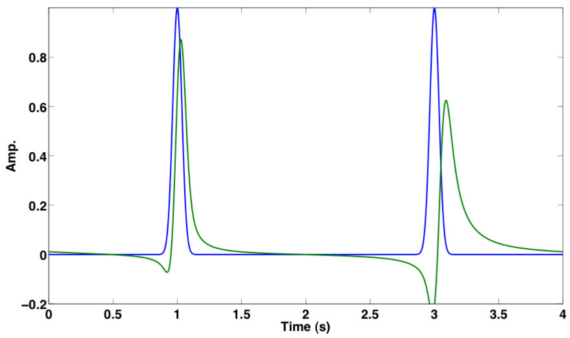


Figure 1. Q-attenuation of seismic impulses.

Figure 1 shows the output of the Gabor numerical model for Q-attenuation applied to two successive pulses (in blue), resulting in two attenuated, broadened output pulses. The increased attenuation and broadening of the second pulse demonstrations the nonstationary character of the attenuating filter, which more accurately represents the physics of Q-attenuation. The asymmetric output is a result of the minimum phase condition for the nonstationary filter.

In the second example, we use a combination of stationary phase shift operators to model wave propagation in two dimensions. For each fixed velocity, we numerically propagate a wavefield by multiplying by the appropriate phase shift in the Fourier domain. For a complex velocity model, we window the regions to areas of local near-constant velocities, and apply the appropriate stationary operator in that region. The combination of these local operators is again a Gabor multiplier, or generalized frame operators, and is certain to be a stable wavefield propagator. Some sample calculations are shown in Figure 2.

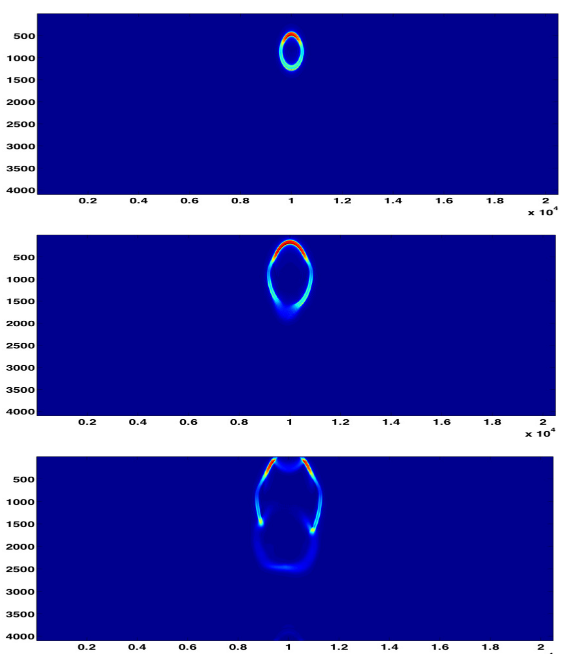


Figure 2. Numerical wavefield propagation

Figure 2 shows the output of three steps of a Gabor multiplier algorithm that numerically propagates a wavefield through a complex velocity model (EAGE salt dome model). Velocities range from 1500 to 4500 m/s and are represented by six Gabor windows with six fixed velocities spanning the range. A fixed velocity propagator is created for each window, the full wavefield is decomposed by each window, propagated one time step, and the various parts recombined before repeating the windowing/propagating step. The three images here show every 20$^{th}$ step of the propagation. We observe the initially circular wavefront becomes distorted as it passes through the various velocity layers of the salt dome model.

## Conclusions

Gabor multipliers, based on localized Fourier transforms and generalized frames, allow the development of effective numerical methods for modeling nonstationary processes in seismic studies. The mathematical theory behind these methods has been developed to provide accurate, informative algorithms for use in deconvolution, attenuation models, and numerical wavefield propagation. These techniques are being used by the CREWES and POTSI research groups as core algorithms in seismic imaging.


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#### References

Asgari, M., and Khosravi, A., 2005, Frames and bases of subspaces in Hilbert spaces: J. Math. Anal. Appl., 308, 541–553.
Casazza, P., and Kutyniok, G., 2004, Frames of subspaces: (Wavelets, Frames and Op. Thy) Contemp. Math., 345, 87–113.
Christensen, O., 2003, An Introduction to Frames and Riesz Bases: Birkhäuser.
Eldar, Y., 2003, Sampling with arbitrary sampling and reconstruction spaces and oblique dual frame vectors: J. Fourier Anal. Appl., 9, No. 77–96.
Fornasier, M., 2003, Decompositions of Hilbert spaces: Local construction of global frames, in Bojanov, B., Ed., Proc. Int. Conf. on Constructive Function Theory Varna, DARBA, Sofia, 271–281.
Fornasier, M., 2004, Quasi-orthogonal decompositions of structured frames: J. Math. Anal. Appl., 289, 180–199.
Lamoureux, M.P., Margrave, G.F., Ismail, S., 2008, Solving physics PDEs using Gabor multipliers, CREWES Ann. Report.
Lamoureux, M.P., Margrave, G.F., Gibson, P.C., 2010, Generalized frames for Gabor operators in seismic imaging, GeoCanada 2010 Abstracts.
Lamoureux, M.P., Margrave, G.F., Gibson, P.C., 2010, Gabor multipliers for pseudodifferential operators and wavefield propagation, CREWES Ann. Report.
Li, S., and Ogawa, H., 2004, Pseudoframes for subspaces with applications: J. Fourier Anal. Appl., 10, 409–431.
Sun, W., 2006, G-frames and G-Riesz bases: J. Math. Anal. Appl., 322, 437–452..
Wards, B.D., Margrave, G.F., Lamoureux, M.P., 2008, Phase-shift time-stepping for reverse-time migration: The Marmousi data experience, CREWES Ann. Report.
Wards, B.D., Margrave, G.F., Lamoureux, M.P., 2008, Fast phase-shift time-stepping for reverse-time migration, CREWES Ann. Report.