A New Migration Method Using a Finite Element and Finite Difference Approach

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Abstract
A new method of migration using Finite Element Method (FEM) and Finite Difference Method (FDM) is jointly used in the spatial domain. It has been applied to solve a time relay 2D wave equation. By using the semi-discretization technique of FEM in the spatial domain, the origin problem can be written as a coupled system of lower dimensions partial differential equations (PDEs) that continuously depend upon time and space. FDM is used to solve these PDEs. The concept and theory of this method are also discussed in this paper. One numerial example of wave-equation migration shows the successful result and its potential application.

1. Introduction

The Finite Element – Finite Difference Method (FE–FDM) is one of the numerical methods using FEM and FDM in the spatial domain to solve partial differential equations. FE-FDM uses FEM in some dimensions and FDM in the remaining dimensions and in the time domain. The FE-FDM has strong resemblance to a number of numerical methods such as the finite difference method and the finite element method. A brief emphasis on the basic differences between FE-FDM and the above mentioned methods is as follows:

FEM fully discretizes a static problem into a system of algebraic equations with discrete nodal values as the basic unknowns. For the time relay problem, FEM fully discretizes it in spatial domain into ordinary diifferential equations (ODEs) and solves them with the FD method (Hughes, 1987), whereas the FE-FDM semi-discretizes the PDE using FEM in the spatial domain into a coupled system of PDEs. These PDEs still continuously depend upon both time and space (although not all the space dimension), and are solved with FD method. Thus, the strengths of FEM, the adaptation to arbitrary domain, boundary, material and loading are retained. The shortcomings of FEM, such as large demand on computer memory and high computation costs are reduced because of the semi-discretization. Compared with the FD method, the computation precision is increased by FEM semi-discretization. The technique of FD for solving PDEs in lower dimensions can decrease frequency dispersion in space and has looser conditions of stability for explicit FD schemes.

In this paper, the basic concept and theory of the finite element – finite difference method are described through the 2-D wave equation. One numerical example is given to demonstrate the improved performance of this method.

2. Principle

Consider the hyperbolic model problem, with the 2-D scalar wave equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial z^2}=\frac{1}{a^2(x,z)}\frac{\partial^2 u}{\partial t^2} \tag{1a}$$

Here $u(x,z,t)$ denotes the wave disturbance at horizontal (lateral) coordinate x, vertical (depth) coordinate z (where the z axis points downward) and time t, respectively. a(x, z) is the medium velocity. We assume a boundary condition (B. C.) of the form:

$$u(x,z,t)=\begin{cases}\varphi(x,t) & z=0\\ 0 & z\neq 0\end{cases},\ \text{in } \partial\Omega \tag{1b}$$

and the initial condition:

$$u(x,z,t=T)=\phi(x,z)\,,\ \dot{u}(x,z,t=T)=0\,,\ \text{in}\,\Omega\,. \tag{1c}$$

The two-dimensional domain Ω is bounded by the piecewise smooth boundary ($\partial\Omega$). The purpose of wave-equation migration is to solve the above equation so that the recorded wave field at t=T can propagate back to t=0; hence the reflected wave lies at the reflection interface (Yilmaz, 1987). FE-FDM discretizes (1a) in the x-coordinate using FEM, and solves the remaining equations in the z-t coordinates using the FD method.

2.1 FEM semi-discretization in x-coordinate

P1 denotes the partial differential equation (1). P2 denotes the corresponding Galerkin method of P1. P2 is:

Find $u\in S^1_\varphi$, such that for all $v\in S^1_0$

$$D(u,v)-F(v)=0 \tag{2}$$

here

$$S_\varphi^1 = \left\{ u \mid \int \left[u^2 + \left(\frac{\partial u}{\partial x}\right)^2 \right] dx < \infty,\ u(x,z,t) = \begin{cases} \varphi(x,t) & z = 0 \\ 0 & z \neq 0 \end{cases}, \text{in } \partial\Omega \right\},$$

$$S_0^1 = \left\{ v \mid \int \left[v^2 + \left(\frac{\partial v}{\partial x}\right)^2 \right] dx < \infty,\ v(\partial\Omega) = 0 \right\},$$

$$D(u,v) = \iint \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} - \frac{1}{a^2(x,z)} \frac{\partial^2 u}{\partial t^2} \right] v dx, \quad F(v) = 0$$

$D(u,v)$ can be rewritten as

$$D(u,v) = \int_\Omega \left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} - \frac{\partial^2 u}{\partial z^2} v + \frac{1}{a^2(x,z)} \frac{\partial^2 u}{\partial t^2} v \right] dx . \tag{3}$$

Semi-discretizing the horizontal coordinate (x) in the region of [0, X], one constructs finite element function space as

$$u_h(x,z,t) = \sum_{i=1}^{N} u_i(t,z) N_i(x), \tag{4a}$$

$$\frac{\partial}{\partial x} u_h(x,z,t) = \sum_{i=1}^{N} u_i(t,z) \frac{d}{dx} N_i(x) = \sum_{i=1}^{N} u_i(t,z) B_i(x), \tag{4b}$$

where N is the nodal numbers of each cell. By substituting equation (3) and (4) into (2), one gets the discrete style description of P2.

$$D(u_h, v_h) = \sum_{n=1}^{NE} \int_\Omega \left[v_e^T B^T B u_e - v_e^T N^T N \frac{\partial^2 u_e}{\partial z^2} + \frac{1}{a^2(x,z)} v_e^T N^T N \frac{\partial^2 u_e}{\partial t^2} \right] dx = 0,$$

NE is the total numbers of all the nodes, u_e, v_e is the each cell vector, e means each cell and this expression can be simplified. For the reason of function v is arbitrary, one obtains semi-discretized PDEs as

$$\mathbf{M} \frac{\partial^2 u}{\partial t^2} + \mathbf{K} u = \mathbf{H} \frac{\partial^2 u}{\partial z^2}, \tag{5a}$$

with boundary condition (B. C.) $u(z=0,t) = g(t)$, (5b)

and initial conditions $u(z, t=T) = f(z)$, $\dot{u}(z, t=T) = 0$ (5c)

where $g(t)$ and $f(z)$ are the discretization of $\varphi(x,t)$ and $\phi(x,z)$ respectively.

$$\mathbf{M} = \sum_{n=1}^{NE} \mathbf{M_e},\ \mathbf{K} = \sum_{n=1}^{NE} \mathbf{K_e},\ \mathbf{H} = \sum_{n=1}^{NE} \mathbf{H_e} \tag{5d}$$

$$\mathbf{M_e} = \int_e \frac{1}{a^2(x,z)} N^T N dx,\ \mathbf{K_e} = \int_e B^T B dx,\ \mathbf{H_e} = \int_e N^T N dx \tag{5e}$$

It can be seen that the matrices **M**, **K** and **H** are all symmetric. **M** and **H** are positive-definite, and **K** is positive-semidefinite. The PDEs are model hyperbolic equations when the velocity is constant because the matrices **M** and **H** can be diagonalized at the same time under this condition. It should be emphasized that only the matrix **M** varies with depth.

2.2 FDM solution of matrix PDEs

One of the explicit schemes, the five-point central scheme, is selected to solve this problem. The difference equation has the form

$$\mathbf{M} u[i]_j^{n-1} = \mathbf{A} u[i]_j^n + \mathbf{B}(u[i]_{j+1}^n + u[i]_{j-1}^n) - \mathbf{M} u[i]_j^{n+1} \tag{6}$$

where $\mathbf{A} = 2\mathbf{M} - \tau^2 \mathbf{K} - \frac{2\tau^2}{h^2}\mathbf{H}$, and $\mathbf{B} = \frac{\tau^2}{h^2}\mathbf{H}$, where τ and *h* are the time and space steps, assumed constant, and *i*, *j*, *k* are the discrete denotation of lateral direction, depth direction and time, respectively. The local truncation error of this scheme has the form of O($\tau^2 + h^2$) (Durran, 1999).

2.3 Stability discussion of matrix PDEs

The stability of a wave equation with a B. C. and I. C. is much complicated. For this problem, the stability is difficult because it is related to the FEM semi-discrete scheme and the form of the interpolation function.

The scheme stability analysis of the simplest condition is discussed here. Considered piecewise linear interpolation function, this problem has been supposed to have only one element. The element length is l, the velocity is a, x_i is the each node coordinate value. The interpolation function (assume n=3, x_1, x_2, x_3 is the three node coordinate value)

$$N(x) = (\frac{1}{2}\xi(1-\xi), 1-\xi^2, \frac{1}{2}\xi(\xi+1)),$$

where $\xi = 2\frac{x_i - x_2}{l}$, $x_2 = \frac{x_1 + x_3}{2}$.

By using the central scheme in both the time and space, we use the Fourier analysis method. It can be obtained as follows, and here τ and h are the time and space steps, and $\lambda = \tau / h$.

$$a^2\lambda^2 + \frac{3\tau^2}{4l^2} \le 1. \qquad (7)$$

It can be verified that the equation (7) is the sufficient and necessary stability condition for the discretatzation scheme . When the velocity of the wave equation is much larger than 3/4, This condition is much looser than that ($a^2\lambda^2 < 1/2$) of the 2-D space and time central scheme FD method (Lu and Guan, 1987). It is to mean that the time-step restriction imposed in FE-FDM is often much smaller than that needed for accuracy in FDM.

3. Numerical examples

In order to test the validity of the FE-FDM, one numerical example of steep obliquity migration with variational velocity is chosen. To show the advantage and potential of FE-FDM, it is compared with the FK-FD migration method.

3.1 Steep oblique model introduction

The model for this section is shown in Fig. 1. The velocity of the model increases both laterally and with depth direction. The velocity at the top left corner is 3600m/s, and the at the bottom right it is 4600m/s. There are four reflection interfaces, those with a decline of 0 degree,23degrees, 45 degrees, and 70 degrees.

The seismograph is computed by the FDM module of the SU Software Kit, and is displayed in Fig. 2. From it, one can see that there are many diffractions because of the incline interfaces. To remove these influences, the FE-FDM and the FK-FDM are used to test the effect of migration. Fig.3 is the result of FE-FDM, and Fig.4 is the result of FK-FDM.

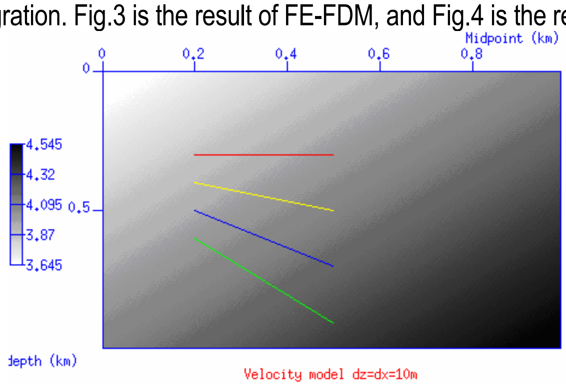

Fig. 1 Steep oblique Model

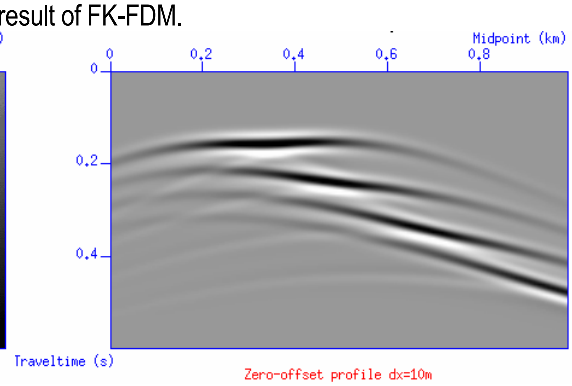

Fig. 2 The seismograph in use of FDM

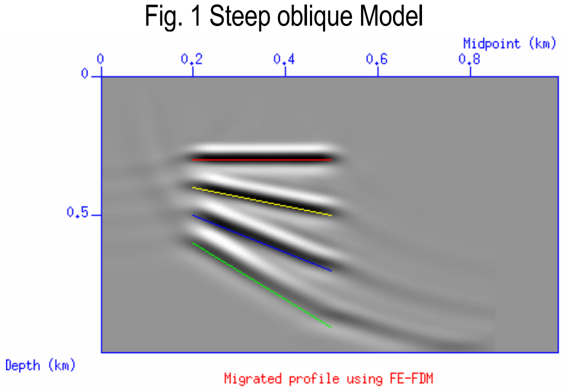

Fig.3 The result of FK-FDM

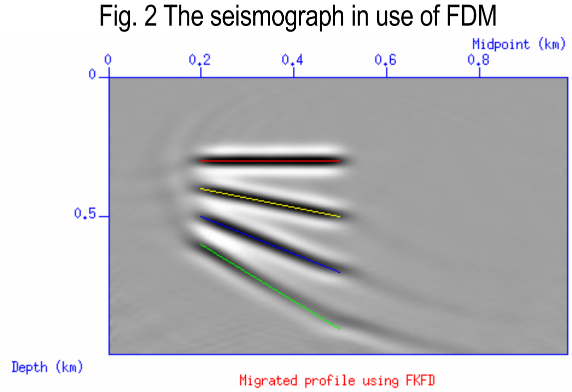

Fig. 4 The result of FE-FDM

3.2 Remark

It can be seen from Fig.3 and Fig.4 that both methods can effectively image the position of layers and have good correspondence with the model. In addition, the two methods can accurately image the geometry under the oblique reflector and the inhomogeneity. In fact, the FEFDM has more advantage on the efficiency. From the Fig. 5 and Fig. 6(one is the contrast of compuation time, and the other represents the used memory), we can know that FEFDM can save almost 6 times as the FKFDM under using the two memories. Therefore, we can get FEFDM is much higher efficient with the higer resolution.

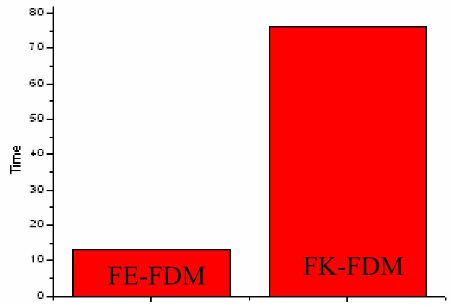

Fig.5 The consumed time of the two methods

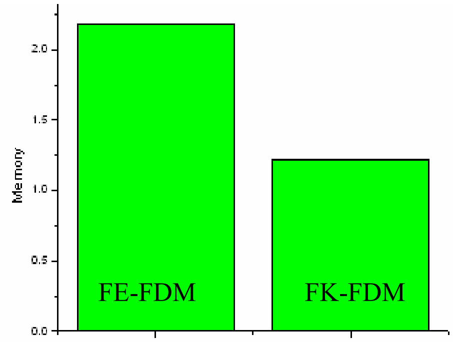

Fig. 6 The consumed memory of the two methods

3.3 Discussions

Omiga-x domain FD migration is a method for the one-way approximation wave equation. The equation used in this paper is accurate for propagation directions to 90 degrees. FT and IFT are applied for time only. The FD method is used in omig-x domain for wavefield extrapolation (Lee and Suh, 1985). This algorithm fits lateral velocity variation and complex interfaces so well that it has become the most popular migration method today. The drawbacks of it are the high computation costs, spatial dispersion, and its limited ability to image steeply dipping interface.

FEM migration is a highly accurate method for the problem of arbitrary shaped domains, lateral velocity variations, and complex and dipping interfaces (Teng and Dai, 1989). But it is not widely used in seismic exploration because of its large demands on computational costs and computer memory. FE-FDM migration inherits all the advantages of FEM migration presented above. The computational efficiency is improved through spatial domain semi-discretization. As shown above, the FE-FDM migration can successfully be applied to field data.

4. Conclusions

A numerical method named finite element–finite difference method (FE-FDM) for the solution of time relay partial differential equations such as parabolic and hyperbolic model equations is presented in this paper. As the numerical examples, 2-D scalar wave equation reverse-time depth migration has been shown above, and it is encouraging that the result is accurate and effective enough for steeply dipping interface imaging.

This method combines FEM and FDM based upon the semi-discretization of the spatial domain. The main strengths of FEM (adaptation to arbitrary domain and accuracy) and FEM (computation efficiency) are inherited. FE-FDM can be used to get accurate results for migration more accurately than the FD method and much more quickly than FEM. At the same time, it can be used to implement elastic-wave equation migration, or simulate wave propagation. It is therefore a useful and promising numerical method.

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