Elastic finite difference modelling with stability and dispersion corrections

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Summary

This paper presents a general method to limit the dispersion and instability inherent within finite-difference elastic modelling in two dimensions. The method is based on an extension of the Von Neumann stability analysis. For a fixed frequency an analytic relationship is derived between the continuous derivatives in the elastic wave equation and their second-order finite-difference approximations. Typically, the continuous derivative is equal to a finite-difference result divided by a correction factor that is a squared sinc function dependant on frequency and grid size. When the continuous derivatives are replaced by these expressions, an exact formulation of the elastic wave equation results that involves finite differences and correction factors. These correction factors are all frequency dependent. The frequency dependence can be converted to wavenumber dependence using P and S wave velocities. This allows the correction factors to be applied as spatial filters. Numerical tests show that these correction factors compensate for a wide range of dispersion and instability.

Introduction

The fact that a simple substitution of finite differences for continuous differentials is unstable or dispersive has long been understood, for example as Von Neumann stability analysis in Aki and Richards (1980). The early studies showed the nature and magnitude of the problem in order to allow an appropriate selection of sample rates. Later papers addressed the economics of solving real problems, for example by using higher order spatial derivatives to allow a more coarse spatial sample rate as in Levander (1988). This report presents another approach to improving the estimate of continuous differentials by the type of analysis presented in Press et al (1992) for stability estimation. It is very similar to, and an extension of, the one spatial dimension study in Manning and Margrave (2000).

Theory

The two-dimensional, continuous, elastic wave equation is (along with a similar equation with x and z switched)

$$(\lambda+2\mu)\frac{\partial^2 U_z}{\partial z^2}+(\lambda+\mu)\frac{\partial^2 U_x}{\partial x\partial z}+\mu\frac{\partial^2 U_z}{\partial x^2}=\rho\frac{\partial^2 U_z}{\partial t^2}\,. \tag{1}$$

Here λ and μ are the Lame elastic parameters for an isotropic medium and U_x and U_z are the components of the particle displacement. It can be shown that:

$$\frac{\partial^2 U_z}{\partial z^2}=\frac{1}{\operatorname{sinc}^2\left(\frac{k_z\Delta z}{2}\right)}D_{\Delta z}{}^2 U_z\,, \tag{2}$$

where D_{Δ_z} is the finite difference version of a differential with respect to z. An example of this type of derivation for one spatial dimension is shown in Manning and Margrave (2000). By substituting this and analogous expressions in equation (1) the following equation is derived:

$$\left\{\begin{array}{l}(\lambda+2\mu)\dfrac{D_{\Delta z}{}^2 U_z}{\operatorname{sinc}^2\left(\frac{k_z\Delta z}{2}\right)}+\mu\dfrac{D_{\Delta x}{}^2 U_z}{\operatorname{sinc}^2\left(\frac{k_x\Delta x}{2}\right)}\\ +(\lambda+\mu)\dfrac{D_{\Delta x\Delta z}U_x}{\operatorname{sinc}\left(\frac{k_x\Delta x}{2}\right)\operatorname{sinc}\left(\frac{k_z\Delta z}{2}\right)}\end{array}\right\}=\rho\frac{\mathrm{D}_{\Delta t}{}^2 U_z}{\operatorname{sinc}^2\left(\frac{\omega\Delta t}{2}\right)}\,. \tag{3}$$

In a time-stepping scheme of wavefield modelling, the left side of equation (3) is fully determined because the complete wavefield in space is available to calculate the spatial wavenumbers in x and z. Therefore the acceleration can be calculated exactly. The sinc correction on the right side of the equation can not be made in a straightforward way because the temporal frequencies are not available until the model is complete.

In the one-dimensional case ω can be determined from the formula ω=vk**,** as in Manning and Margrave (2000). This works because there is only one velocity and it can be used to relate the spatial and temporal frequencies. The frequency correction is then turned into a wavenumber correction and the deterministic time step is easily calculated.

A similar approach can be used for the two-dimensional case. Correction factors for the $D_{\Delta_z}U_z$ and $D_{\Delta_x}U_z$ terms can be determined by considering the P and S waves that are affected only by these terms. The $D_{\Delta_x\Delta_z}U_x$ correction can then be chosen to allow consistent wave propagation in other directions. The following equation is then obtained:

$$\left\{ (\lambda+2\mu)\frac{\mathrm{sinc}^2\left(\frac{v_\alpha k\Delta t}{2}\right)}{\mathrm{sinc}^2\left(\frac{k_z\Delta z}{2}\right)} D_{\Delta z}{}^2 U_z + \mu\frac{\mathrm{sinc}^2\left(\frac{v_\beta k\Delta t}{2}\right)}{\mathrm{sinc}^2\left(\frac{k_x\Delta x}{2}\right)} D_{\Delta x}{}^2 U_z + \frac{(\lambda+2\mu)\mathrm{sinc}^2\left(\frac{v_\alpha k\Delta t}{2}\right) - \mu\mathrm{sinc}^2\left(\frac{v_\beta k\Delta t}{2}\right)}{\mathrm{sinc}\left(\frac{k_x\Delta x}{2}\right)\mathrm{sinc}\left(\frac{k_z\Delta z}{2}\right)} D_{\Delta x\Delta z} U_x \right\} = \rho \mathrm{D}_{\Delta t}{}^2 U_z \, . \qquad (4)$$

It is most fortunate that when the three correction factors shown above are chosen, no further corrections are required. Both P and S waves propagate at any angle with a perfect simulation of wave propagation in a continuous constant velocity medium. Each additive term consists of a product of elastic constants, a finite difference operator, and a ratio of sinc functions. We refer to these various sinc function ratios as ‘correction factors’. If all of the sinc functions are set to unity, then the conventional, second order, finite difference elastic wave equation results. If the sinc functions are directly evaluated, they ‘correct’ the finite difference for its dispersion and instability. However, the correction factors are wavenumber dependent and so must be applied as spatial filters

Applications

The method described here has been tested by a straightforward Fourier domain approach. For each component, the result of each of the three finite difference terms were transformed into the Fourier domain and multiplied by the appropriate wavenumber surface. The components were then added together and inverse transformed for use in time stepping. Thus the correction factors are applied after each time step.

An example of a wavenumber surface for the first term in equation (4) is shown in Figure 1. The sinc function ratio corrects for a net dispersion effect in the z direction, and a net unstable effect in the x direction. Figure 2 shows one quarter of the same correction operator in space (it is symmetric in x and z).

A comparison of uncorrected and corrected modelling is shown in Figures 3 and 4. Note the 'square' shape of the uncorrected modelling compared to the corrected. Figures 5 and 6 show uncorrected and corrected modelling of cylindrical P and S waves at half the sampling rates used for the previous case. The pressure wave has been propagated by the uncorrected modelling in Figure 5 much better than in Figure 3 because of the finer sample rate. However a close inspection of the uncorrected pressure wave in Figure 5 shows that it hasn't preserved the zero-phase nature of the wavelet evident in the corrected version of Figure 6. The corrected S wave in Figure 6 is obviously much improved.

Conclusions

The continuous elastic wave equation in two dimensions can be reformulated into a finite difference equation, including correction factors, which is theoretically exact. Since the continuous solutions have no dispersion or instability, the equivalent finite difference solutions should be stable and non-dispersive also. More general waves can be constructed from plane waves, and they too should be propagated correctly.

Application of the theoretical corrections to practical models has confirmed the stable and non-dispersive properties. The corrections were applied in the wavenumber domain after Fourier transforms, and even though no special coding was used to suppress edge effects or wraparound, the results were satisfactory for the basic stable case.

Acknowledgements

We would like to thank the sponsors of CREWES for their generous support.

References

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Levander, A. R., 1988, Geophysics, Vol. 53, No. 11, P. 1425-1436.
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Press, W. H., Teukolsky, S. A., Vetterling, W. T., & Flannery, B. P., 1992, Numerical Recipes in C: Cambridge University Press.

Figures

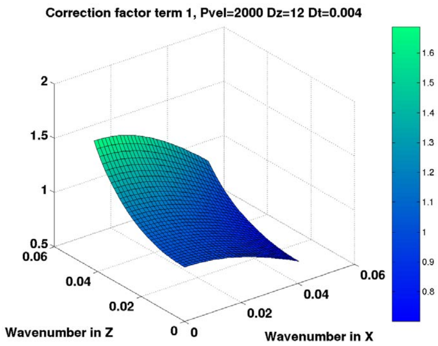

Figure 1. The correction factor for term 1 in equation (4) is shown in the wavenumber domain

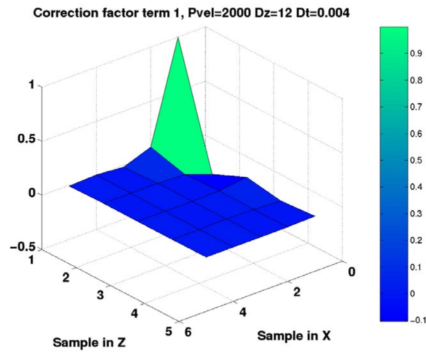

Figure 2. The correction factor for term 1 in equation (4) is shown in space.

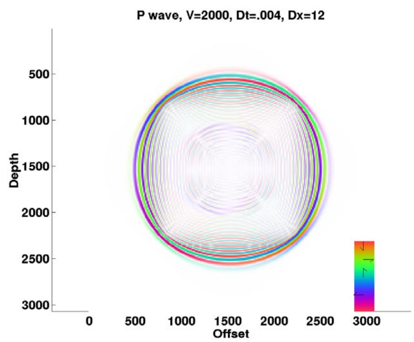

Figure 3. An uncorrected P wave is shown after propagating 120 time steps from a source at 1500, 1500. Note the 'square' shape to the dispersion pattern within the wavefront.

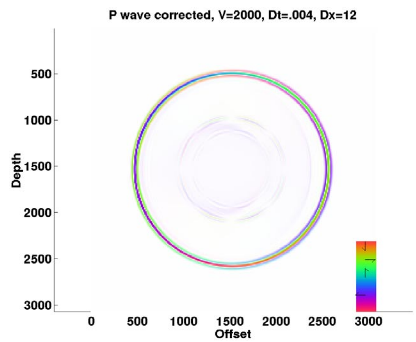

Figure 4. This is similar to Figure 3 except that the correction factors have been applied after each time step.

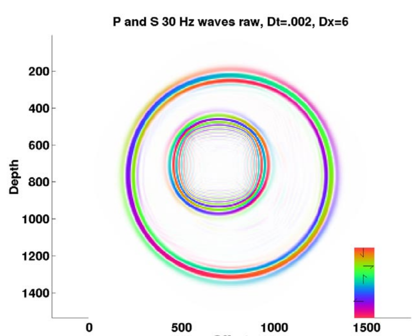

Figure 5. The uncorrected model propagated 120 steps from an initial P source (large) and S source (small) is shown. Note that the scale has changed. The finer sample rate in space and time compared with Figure 3 has allowed the P wave to propagate quite evenly, but the S wave remains distorted.

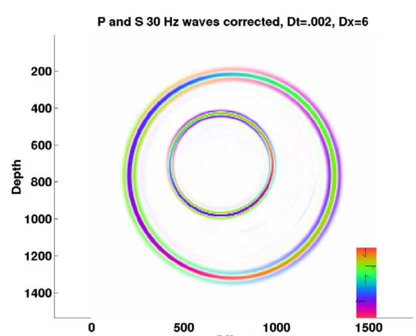

Figure 6. This shows the corrected model propagation from the same sources as Figure 5. The S wave has retained its circular shape and the P wave has retained its zero-phase character.