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SPECULATIONS ON BIOT "SLOW" WAVES

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ABSTRACT

Biot waves are elastic waves that propagate in fluid-saturated porous media. Such waves are of interest to us because they relate to seismoelectric exploration of near-surface materials, a major research interest of our laboratory. In order to obtain a better intuitive understanding of the phenomenon, we have developed a formulation that is based on a highly intuitive physical model. The premise of this article is that the essential form of the result and characteristics of the appropriate wave equations can be very simply derived, with a minimum of physical assumptions. Though less comprehensive than the usual derivations, the procedure proposed has the virtue that the algebra is simpler and the formulation more directly related to the physical process. In terms of the form of the result, nothing of value is lost.

One consequence of reexamining the theory has been the discovery that slow waves cannot be produced in the absence of dissipation, as has been claimed in some published papers. Root locus diagrams are used to examine the nature of the waves in viscous materials.

INTRODUCTION

In a porous, fluid-saturated solid, both solid and fluid phases are capable of transmitting compressional waves. If these waves were uncoupled, the analysis would be trivial – there would be two modes with velocities equal to the intrinsic velocities of the solid and fluid. However, coupling does take place, both through inertial and viscous forces. Solutions for the resulting waves must properly take this coupling into account. The most widely quoted references for the propagation of such waves are those of Biot (1956, 1962), whose research was based on the earlier work of Frenkel (1944).

Our laboratory has been carrying out research into seismoelectric prospecting methods, in which a seismic source is used to excite electromagnetic responses in suitable targets. There are several mechanisms for such conversion, including piezoelectricity, modulation of the resistivity of the ground through which telluric currents pass, stimulation of high-frequency electromagnetic responses from certain sulphides and electrokinetic effects. The latter are thought to be related to Biot waves (see, for example, Butler et al., 1994, in press).

In general, there is relative motion between fluid and solid. Depending on the electrochemical effects at the fluid-solid interface, the net charge on the two phases can be different from zero and of opposite sign. In this case, there will be electrical phenomena as a byproduct of the wave propagation. Although it is the electrical effects that are of direct interest for our seismoelectric research, these will not be considered here. Most of the present paper is based directly on Tolstoy (1973) who gives a streamlined view of the 1962 Biot paper.

Use of Lagrange's equations is a widely favoured technique for solving physical problems that involve coupled oscillations or waves. It is argued, for example by Tolstoy (1973), that such a strategy is economical because one technique can be applied to a large number of mechanical (and electromechanical) problems. In fact, earlier authors who have derived forms of the Biot equations generally have done so with at least some reference to Lagrange's equations.

The most familiar form of Lagrange's equation is for oscillating systems in which there is evaluated a Lagrangian function (sometimes called the "action") defined by $L \equiv T - V$, where T is the total kinetic energy of the system and V is the total potential energy. Both energies, and therefore also L, are functions of a number of generalized coordinates $u^{(i)}$ and of their derivatives. The generalized coordinates are chosen so that they can be independently varied without violating the constraints of the system and so that together they are just sufficient to describe the configuration of the system. In addition, they may be explicit variables of time, t. Lagrange's equations, which can be easily derived from Hamilton's principle, are then a series of differential equations (one for each $u^{(i)}$) having the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial u_t^{(i)}}\right) - \left(\frac{\partial L}{\partial u^{(i)}}\right) = 0. \qquad (1)$$

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In the notation used, the superscript in parentheses plays the role of an index and a subscript indicates partial differentiation.

In the case of wave propagation in one dimension, the physical description of the process is a function of position as well as of time, and the above equation must be extended to include position variables x. This involves two adjustments. The first is to redefine the energies and the Lagrangian function so that they represent energy *densities* at a particular point. The second is to add terms to equation (1) to accommodate external generalized forces (i.e., those not accounted for by V) and dissipative terms. The result is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial u_t^{(i)}}\right)+\frac{d}{dx}\left(\frac{\partial L}{u_x^{(i)}}\right)-\frac{\partial L}{\partial u^{(i)}}=f^{(i)}-\frac{\partial D}{\partial u_t^{(i)}}.$$

The quantities T, V and L are redefined to be energy densities; D is a dissipation term which, for simple linear systems, is twice the power dissipated per unit volume (per unit length in the case of one-dimensional problems).

The Viscosity-free Solution

Simplify the problem by ignoring shear waves and by assuming one-dimensional wave propagation. Except at interfaces, shear waves are not coupled to compressional waves. The Biot and Tolstoy formulations, which provide for shear waves, show that they are uncoupled to the waves of interest here.

Assume a porous medium that is isotropic and macroscopically homogeneous (the grain size is much smaller than the seismic wavelength). Consider a long prism cut out of such material with axis parallel to the x-axis, the direction of propagation. We make the following assumptions:

- A typical cross-section of the prism is made up of three recognizable phases of which the proportions are given by:
  θ_s = fraction of solid;
  θ_f = fraction of fluid, uncoupled to the solid;
  θ_b = fraction of boundary layer, coupled to both.
- The density of the solid is ρ_s and of the fluid is ρ_f; the appropriate elastic constant of the solid is E_s and of the fluid is E_f. The physical properties of the boundary layer are those of the fluid.
- As discussed above, ignore shear waves which will not be coupled to compressional waves except at interfaces.
- $u^{(s)}(x,t)$ is the displacement of the solid phase, $u^{(f)}(x,t)$ is the displacement of the fluid phase and $u^{(b)}(x,t)$ is the displacement of the boundary layer.
- The kinetic energy density T is given by

$$2T=\rho_s\theta_s\left(u_t^{(s)}\right)^2+\rho_f\theta_f\left(u_t^{(f)}\right)^2+\rho_f\theta_b\left(u_t^{(b)}\right)^2. \tag{2}$$

- The potential energy density V is given by

$$2V=E_s\theta_s\left(u_x^{(s)}\right)^2+E_f\theta_f\left(u_x^{(f)}\right)^2+E_b\theta_b\left(u_x^{(b)}\right)^2. \tag{3}$$

- The displacement of the boundary layer is the mean of the displacements of the fluid and solid phases: $u^{(b)} = (u^{(s)} + u^{(f)})/2$. If details of the pore configuration are known, then the factor 2 can be replaced by a better figure, but the *form* of the solution should not be changed.

After substituting for $u^{(b)}$, apply Lagrange's equations, taking in turn for the generalized coordinates $u^{(s)}$ and $u^{(f)}$. After some simplification we are left with

$$\left(\rho_s\theta_s+\frac{\rho_f\theta_b}{4}\right)\frac{\partial^2u^{(s)}}{\partial t^2}+\frac{\rho_f\theta_b}{4}\frac{\partial^2u^{(f)}}{\partial t^2}=\left(E_s\theta_s+\frac{E_f\theta_b}{4}\right)\frac{\partial^2u^{(s)}}{\partial x^2}+\frac{E_f\theta_b}{4}\frac{\partial^2u^{(f)}}{\partial x^2}$$

$$\frac{\rho_f\theta_b}{4}\frac{\partial^2u^{(s)}}{\partial t^2}+\left(\rho_f\theta_f+\frac{\rho_f\theta_b}{4}\right)\frac{\partial^2u^{(f)}}{\partial t^2}=\frac{E_f\theta_b}{4}\frac{\partial^2u^{(s)}}{\partial x^2}+\left(E_f\theta_f+\frac{E_f\theta_b}{4}\right)\frac{\partial^2u^{(f)}}{\partial x^2}.$$

For easier comparison with Tolstoy and Biot, we may rewrite the last two equations in the following form which parallels that of Tolstoy's equations (5-63) and (5-64):

$$\rho_{11}\frac{\partial^2u^{(s)}}{\partial t^2}+\rho_{12}\frac{\partial^2u^{(f)}}{\partial t^2}=E_{11}\frac{\partial^2u^{(s)}}{\partial x^2}+E_{12}\frac{\partial^2u^{(f)}}{\partial x^2} \tag{4}$$

$$\rho_{12}\frac{\partial^2u^{(s)}}{\partial t^2}+\rho_{22}\frac{\partial^2u^{(f)}}{\partial t^2}=E_{12}\frac{\partial^2u^{(s)}}{\partial x^2}+E_{22}\frac{\partial^2u^{(f)}}{\partial x^2}, \tag{5}$$

where

$$\rho_{11}\equiv\rho_s\theta_s+\rho_f\theta_b/4 \qquad E_{11}\equiv E_s\theta_s+E_f\theta_b/4$$

$$\rho_{12}\equiv\rho_f\theta_b/4 \qquad E_{12}\equiv E_f\theta_b/4$$

$$\rho_{22}\equiv\rho_f\theta_f+\rho_f\theta_b/4 \qquad E_{22}\equiv E_f\theta_f+E_f\theta_b/4.$$

Tolstoy used κ for E_{11}, R for E_{22} and Q for E_{12}.

From these equations, the velocity of the compressional waves propagating in the solid and (uncoupled) fluid phases can be directly calculated as follows. Take the two-dimensional Fourier transform of equations (4) and (5). Using the symbols ω for the angular frequency in radians per second and k for 2π times the wavenumber, there results immediately the matrix equation

$$-\omega^2\begin{pmatrix}\rho_{11} & \rho_{12}\\ \rho_{12} & \rho_{22}\end{pmatrix}\begin{pmatrix}u^{(s)}\\ u^{(f)}\end{pmatrix}=-k^2\begin{pmatrix}E_{11} & E_{12}\\ E_{12} & E_{22}\end{pmatrix}\begin{pmatrix}u^{(s)}\\ u^{(f)}\end{pmatrix}.$$

Substituting v for ω/k, where v is the velocity, there results

$$\begin{pmatrix}v^2\rho_{11}-E_{11} & v^2\rho_{12}-E_{12}\\ v^2\rho_{12}-E_{12} & v^2\rho_{22}-E_{22}\end{pmatrix}\begin{pmatrix}u^{(s)}\\ u^{(f)}\end{pmatrix}=0. \tag{6}$$

Equation (6) yields nontrivial solutions only if the determinant of the coefficient matrix is zero. This condition results in the quadratic equation

$$\left(v^2\rho_{11}-E_{11}\right)\left(v^2\rho_{22}-E_{22}\right)-\left(v^2\rho_{12}-E_{12}\right)^2=0, \tag{7}$$

comparable to Tolstoy's equation (5-74). It correctly predicts four velocities, one for each of the coupled waves in each of the positive and negative directions.

Substituting for ρ_{ij} and E_{ij} in equation (7) and simplifying results in expressions for two velocities. The first is given by

$$v = \sqrt{\frac{E_f}{\rho_f}} \quad ,$$

which is the velocity of the fluid phase, alone. Recognizing that the porosity $\phi = \theta_f + \theta_b$ and that $1 - \phi = \theta_s$, it is possible to define a parameter

$$\theta = \frac{1-\phi}{\phi}\left(\frac{4-3\theta_b/\phi}{(\theta_b/\phi)(1-\theta_b/\phi)}\right)$$

in terms of which the second velocity is

$$v = \sqrt{\frac{\theta E_s + E_f}{\theta \rho_s + \rho_f}},$$

which lies intermediate between the solid and fluid velocities. The factor θ cannot be very small; it is infinite for $\theta_b = 0$ and for $\theta_b = \phi$ and has a minimum value of $9(1-\phi)/\phi$ when $\theta_b = 2\phi/3$. Therefore, the second velocity is always a real number intermediate between the velocities of the solid and fluid, but generally much closer to the velocity of the solid. For the two extremes $\theta_b = 0$ and $\theta_b = \phi$ the two velocities are simply the velocities appropriate to the solid and fluid phases, alone. Otherwise, the velocity of the fluid phase is the velocity of the fluid alone, and there is a minimum value for the velocity in the solid phase which is never very far from the inherent velocity of the solid. Irrespective of the value of θ_b, the velocities are real, as would be expected for a system without dissipation.

While the agreement with the form of Tolstoy's equations is superficially perfect, we have not found it possible to reconcile the numerical values for the velocities. Let us briefly review the differences in our approaches. Rather than starting with a physical model as has been done here, Tolstoy required that the process be linear in which case the expressions for the energies are quadratic forms. While this is an elegant approach, it gives no indication of the signs of the cross coefficients ρ_{12} and E_{12}. Whereas those terms in the present development are clearly positive, Tolstoy made use of conceptual experiments to show that his ρ_{12} is negative. These conceptual experiments consisted of evaluating the sign of forces required to constrain the fluid to be stationary, and seem irrefutable. Tolstoy did not discuss the sign of E_{12}.

The difference of signs can be directly traced to the choice of the displacement variables. Tolstoy made the same choice as was made here for $u^{(s)}$ but for the second displacement he used a representative velocity for the entire fluid including the coupled portion. He provided coupling by assuming that part of the fluid was carried along with the solid. Our displacements are related to Tolstoy's displacements by simple linear equations. If we make such appropriate substitutions, there result new equations in which ρ_{12} is negative. This would give perfect agreement with Tolstoy except for the fact that the signs of ρ_{12} and E_{12} are always found to be the same. Tolstoy substituted negative numerical values for ρ_{12} and positive values for E_{12}. This error seems to explain the difference between our conclusions.

Although it is disconcerting that the velocities depend on the choice of the independent variables, one can always define uncoupled normal modes from which all other modes can be found by linear combination. In fact, the velocities corresponding to the variables as defined in this paper are the velocities of the normal modes. Equation (4) and (5) can be written

$$\begin{pmatrix} u^{(s)} \\ u^{(f)} \end{pmatrix} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{12} & \rho_{11} \end{pmatrix}^{-1} \begin{pmatrix} E_{11} & E_{12} \\ E_{12} & E_{11} \end{pmatrix} \begin{pmatrix} u^{(s)} \\ u^{(f)} \end{pmatrix}. \qquad (8)$$

The eigenvalues of the product of the 2 x 2 matrices in this equation are the velocities of the normal modes and are independent of the choice of variables. It turns out that the velocities of the normal modes are the velocities evaluated above with the conventions of this paper and therefore we need proceed no further.

The conclusion of this section is that inertial forces alone, although they do perturb the velocities characteristic of the individual phases of a porous solid, are insufficient to cause major effects. In order to substantially change the velocities of the waves through porous material we have to reconsider the decision to ignore viscosity.

EFFECT OF VISCOSITY

To this point, the effect of viscosity has been ignored. To include viscous effects, the term D must be included in the Lagrange formulation. Staying with the definitions of variables used throughout this paper, we might suppose that the power dissipated would be proportional to the square of the velocity of the fluid relative to the solid, $D = \kappa\,(u_t^{(s)} - u_t^{(f)})^2$; κ is proportional to the viscosity for non-turbulent flow. This is exactly the form adopted by Biot (1962), who took our constant κ to be the viscosity divided by the coefficient in Darcy's law. While it is not difficult to add the viscous effects to the equations already derived, the result is cumbersome and we choose to drop the inertial coupling terms which have been shown above to be of minor consequence.

Applying Lagrange's equation with the above expression for D gives for the case $\theta_b = 0$,

$$\begin{pmatrix} \theta_f \rho_f & 0 \\ 0 & \theta_s \rho_s \end{pmatrix} \begin{pmatrix} u_{tt}^{(f)} \\ u_{tt}^{(s)} \end{pmatrix} + \begin{pmatrix} \kappa & -\kappa \\ -\kappa & \kappa \end{pmatrix} \begin{pmatrix} u_t^{(f)} \\ u_t^{(s)} \end{pmatrix} = \begin{pmatrix} \theta_f E_f & 0 \\ 0 & \theta_s E_s \end{pmatrix} \begin{pmatrix} u_{xx}^{(f)} \\ u_{xx}^{(s)} \end{pmatrix}.$$

For the case of sinusoidal variations with frequency ω, this can be written:

$$\begin{pmatrix} \rho_f / E_f + \kappa/(j\omega\theta_f E_f) & -\kappa/(j\omega\theta_f E_f) \\ -\kappa/(j\omega\theta_f E_f) & \rho_s / E_s + \kappa/(j\omega\theta_f E_f) \end{pmatrix} \begin{pmatrix} u_{tt}^{(f)} \\ u_{tt}^{(s)} \end{pmatrix} = \begin{pmatrix} u_{xx}^{(f)} \\ u_{xx}^{(s)} \end{pmatrix}.$$

The eigenvalues of the 2 x 2 matrix in the above expression give the reciprocals of the squares of the velocities of the normal modes (i.e., the slowness is the square root of an eigenvalue). Representing an eigenvalue by λ there results the equation:

$$\lambda^2 - \lambda(\rho_f/E_f + \rho_s/E_s) + (\rho_f\rho_s)/(E_fE_s) = (\kappa/j\omega)\{\lambda/(\theta_fE_f) + \lambda/(\theta_sE_s) - (\rho_f/\theta_s + \rho_s/\theta_f)/(E_fE_s)\}. \quad (9)$$

In order to obtain an intuitive understanding of the behaviour of the eigenvalues of this equation, and hence of the normal mode slownesses $\pm\sqrt{\lambda}$, it is helpful to construct the locus of the roots in the complex plane. Fortunately, the equation is almost in the form used in servomechanism theory for root locus solutions (Evans, 1954; Truxall, 1955). That method is used to estimate the behaviour of s in the equation

$$-1 = K\frac{(s+z_1)(s+z_2)\ldots(s+z_n)}{(s+p_1)(s+p_2)\ldots(s+p_m)}.$$

Typically, the right-hand side of the above equation is the open-loop transfer function of a feedback system, with the real number K proportional to the gain; the z_i are the zeros and the p_i are the poles of the open-loop transfer function. The poles and zeros may be zero, real or complex.

Equation (9) can be put into this form by rearranging and squaring so that

$$-1 = K\frac{(\lambda - e)^2}{(\lambda - a)^2(\lambda - b)^2}, \quad (10)$$

in which

$$K = (\kappa^2/\omega^2)\left(\frac{1}{\theta_fE_f} + \frac{1}{\theta_sE_s}\right),$$

$a \equiv \rho_f/E_f = 1/v_f^2$ the square of the slowness of the *P*-wave of pure fluid,

$b \equiv \rho_s/E_s = 1/v_s^2$ the square of the slowness of the *P*-wave of the pure solid,

$c \equiv \dfrac{\theta_f\rho_f + \theta_s\rho_s}{\theta_sE_s + \theta_fE_f}$ the square of the slowness of the "frozen" mix.

Note that squaring introduces spurious roots. These spurious roots can be identified because they are associated with exponential growth of the waves with time, which is non-physical.

Savant (1958, pages 81-123) gives seven rules for sketching root locus diagrams. With the aid of these, and by hand calculating a few values for the roots, it was possible to construct typical root locus diagrams corresponding to equation (9). The rules of Savant that were used were the following:

Rule 1: *Continuous curves, which comprise the locus, start at each pole ... for $K = 0$. The branches of the locus, which are single-valued functions of gain, terminate on the zeros ... for $K = \infty$.* In the present case the squares of the slowness start on a and b for zero viscosity. One pair of roots ends on c for infinite viscosity, while the second pair is at infinity.

Rule 2: *The locus exists along the real axis where an odd number of poles plus zeros is found to the right of the point.* Since the zeros and poles in equation (10) all occur in pairs, the locus never follows the real axis. Therefore, except at zero or infinite viscosity, the square of the slowness is complex and there is dissipation.

Rule 3: *For large values of gain the locus is asymptotic to the angles ((2k + 1)/(#P – #Z)) 180 deg, where k = 0, 1, 2, ... , #P is the number of poles and #Z is the number of zeros.* For equation (10) there are four poles and two zeros, and therefore the locus for large viscosity has branches that are asymptotic to 90 and 270 deg.

Rule 4: *The starting point for the asymptotic lines is given by (Σ poles – Σ zeros)/(#P – #Z) which is called the centre of gravity of the roots.* For equations with real coefficients, the centre of gravity will always be on the real axis. For equation (10) the centre of gravity is $(2a + 2b - 2c)/2 = a + b - c$.

Rule 6: *Two roots leave or strike the breakaway point at* ±90 deg. The breakaway points are points at which the root locus departs from, or arrives at, the real axis. For equation (10), the breakaway points are at the poles and zeros.

Figure 1 shows a plot of the root locus diagram for equation (10) for the case in which $c = (3b + a)/4$ and $a > b$. Figure 2 shows the plot for the degenerate case in which $c = (b + a)/2$. The vector from the origin to a point on the locus represents the (complex) value of λ, the square of the slowness. Thus, we have immediately that for $K = 0$, λ is real and therefore there is no dissipation. Furthermore, from the position of the poles we recognize that the two eigen velocities are just those of the fluid and solid phases. This is what we could expect, for there is no interaction between the waves carried by the phases. Since the poles and zeros here occur in pairs, there is no point other than the actual poles and zeros where the velocities are real: both waves are damped except for zero and infinite viscosity. For $\lambda = \infty$ there is only one finite velocity. Its value is given by the position of the zero, and its slowness is given by the square root of c, the slowness that is appropriate to the case in which the fluid is "frozen" to the solid so that there is no relative motion. It is real as expected: there is no attenuation because there is no relative motion between the phases.

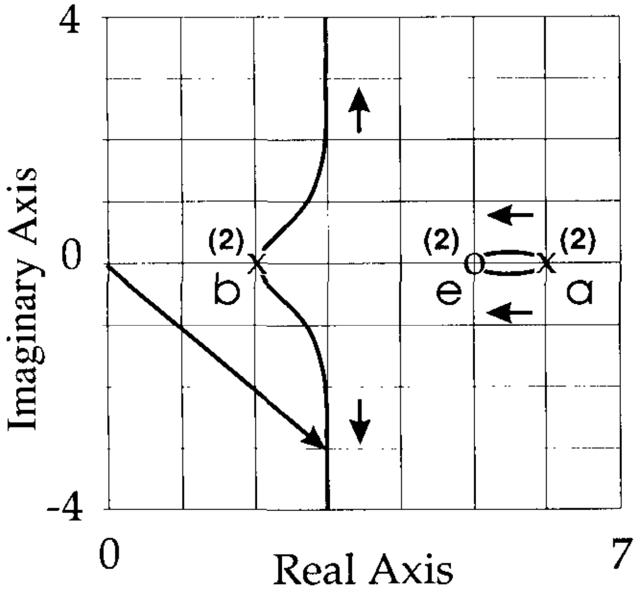

Fig. 1. Example of root locus diagram for the complex slowness of the coupled waves. Here, the poles are at $a = 6$ and $b = 2$ and the zeros are at $e = 5$. The shorter arrows indicate the direction of increasing K. The long arrow points to the square of the complex slowness for a particular value of K. The number two in parentheses, (2), indicates a double pole or zero.

The remainder of this discussion refers in particular to Figure 1; similar arguments apply to Figure 2. For finite values of the viscosity the roots are complex and both waves are dissipative. However, the one wave moves towards the "frozen" value with relatively little change in slowness and small dissipation. The other wave (the "slow" wave) increases rapidly in slowness as the viscosity increases (the point on the locus moves rapidly away from the positive real axis). Both the real and imaginary parts of the slowness increase rapidly without bound; in the limit the slowness and the damping both become infinite.

This behaviour is just what one would expect intuitively, and we conclude that a simple viscosity model leads directly to the expected behaviour of the "slow" wave.

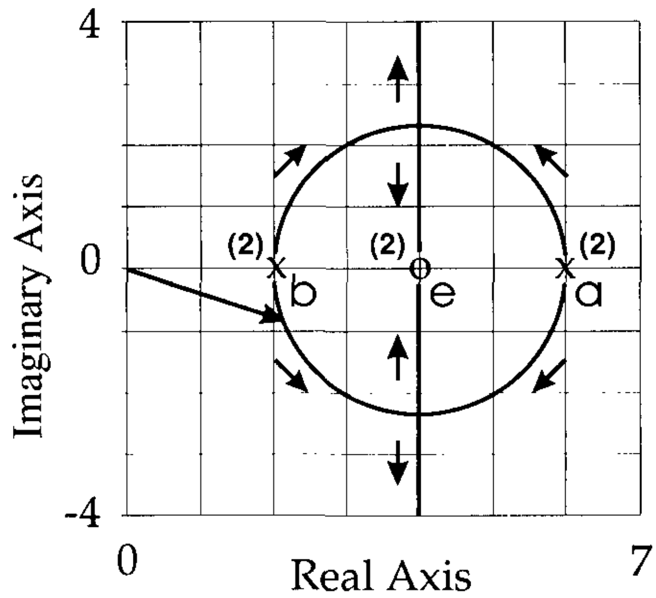

Fig. 2. Example of root locus diagram for the complex slowness of the coupled waves. Here the poles are at $a = 6$ and $b = 2$ and the zeros are at $e = 4$. The shorter arrows indicate the direction of increasing K. The long arrow points to the square of the complex slowness for a particular value of K. The number two in parentheses, (2), indicates a double pole or zero.

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