AN EFFICIENT COMPUTATION OF THE ANGLE OF LATITUDE†

P. LANCASTER*

Introduction

A problem which frequently arises in surveying and geodesy is the calculation of the angle of latitude ϕ from a measured meridional arc m. With a spheroidal model of the earth, a relationship between the two is

$$m = F(\phi) = A_0\phi - 1/2(A_2 \sin 2\phi - A_4 \sin 4\phi + \ldots), \quad \ldots(1)$$

where $A_0, A_2, A_4, \cdots$ are known constants which decrease rapidly in magnitude by a factor of about 10^{-3} from one to the next (Bomford, 1973, Schmid, 1971). Thus, computation of m if ϕ is given presents no problem. Consider the converse problem: Given m, find ϕ We are going to show how this can be done to high accuracy at little more expense than one evaluation of F and one of F', the derivative of F with respect to ϕ

This note is written in the belief that the calculation at hand is one that must be repeated very frequently in some organizations so that an efficient programmable algorithm is desirable. An algorithm is proposed as a candidate for this purpose.

Ostrowski's Theorem

We shall use a modern theorem relating to Newton's iterative method for the solution of $f(\phi) = 0$ where we define $f(\phi) = F(\phi) - m$. In numerical practice an iterative process is often run until iterates "appear" to agree to the desired accuracy and the last estimate is then claimed to be good to the number of digits repeated in the last two steps. This is generally (but not always) an adequate criterion but, as in this case, it may be expensive. We use a theorem of Ostrowski (1973) which allows us to say *with certainty* how close we are to the solution θ of $f(\phi) = 0$ and, as it turns out, we do not have to repeat (iterate) the calculations at all. One application of the Newton formula (2) is all that is required for most practical purposes. This compares with the conclusion of Schmid (1971), for example, that "two or three iterations should be sufficient . . .".

Any text on numerical analysis (and many others) will contain a description of Newton's method. The algorithm is, given an initial estimate ϕ_0 for the solution of $f(\phi) = 0$, compute $\phi_1, \phi_2, \cdots$ recursively from

$$\phi_{n+1} = \phi_n - \frac{f(\phi_n)}{f'(\phi_n)}, \quad n = 0,1,2,\ldots \quad \ldots(2)$$

One then hopes that the ϕ_n's converge to a number θ for which $f(\theta) = 0$.

A simplified version of the Ostrowski theorem cited says:

Define $h_0 = -f(\phi_0)/f'(\phi_0)$ *(the first correction) and let* M *be a number not less than* $|f''(\phi)|$, *where* ϕ *can take any real value. If*

$$2|h_0|M \le f'(\phi_0), \quad \ldots(3)$$

then the ϕ_n*'s converge to a number* θ *for which* $f(\theta) = 0$ *and*

$$|\phi_1 - \theta| \le \frac{M}{|f'(\phi_1)|} h_0^2 \cdot \quad \ldots(4)$$

Note that ϕ_1 is defined by putting $n = 0$ in equation (2) and that the better the approximation ϕ_0 is for θ the smaller h_0 will be, and the smaller the bound on the right of equation (4) will be.

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*University of Calgary, Calgary, Alberta, Canada.

Applying the theorem to our problem and using known values for the A_i coefficients in equation (1) it is found that we may take $M = 6.5 \times 10^4$. Furthermore,

$$|f'(\phi)| = |F'(\phi)| > 6.3 \times 10^6$$

for *all* ϕ We then consider two cases based on different initial estimates ϕ_0.

Case i If we consider only the first term on the right of equation (1) we get an initial estimate $\phi_0 = m/A_0$. Using $f(\phi_0) = F(\phi_0) - m$ we find very easily that $|h_0| \leq 2.6 \times 10^{-3}$. Thus, the condition (3) is certainly satisfied and equation (4) yields

$$|\phi_1 - \Theta| < \frac{6.5 \times 10^4}{6.3 \times 10^6} (2.6)^2 \times 10^{-6} < 7 \times 10^{-8}.$$

Thus, we conclude that with this choice of ϕ_0, the estimate ϕ_1 determined by *one* step of the Newton method differs from Θ by less than seven in the *eighth* decimal place.

If higher accuracy is needed we could repeat the calculation using ϕ_1 in place of ϕ_0. Instead, we propose a cheaper method based on a more accurate initial estimate of ϕ_0.

Case ii If we retain the first two terms on the right of equation (1) and take advantage of the estimate used in Case i the following refined estimate suggests itself: $\phi_0 = \frac{m}{A_0} + \frac{1}{2} \frac{A_2}{A_0} \sin \frac{2m}{A_0}$.

In this case it is found that $|h_0| < (1.6) \times 10^{-5}$ so that

$$|\phi_1 - \Theta| < \frac{6.5 \times 10^4}{6.3 \times 10^6} (1.6)^2 \times 10^{-10} < 2.7 \times 10^{-12}$$

Thus, with a little extra computational expense in obtaining ϕ_0, we obtain a ϕ_1 differing from the true solution by less than three in the *twelfth* decimal place.

Conclusion

Two cautionary remarks may be made. First, there is no point in seeking greater accuracy for the computed latitude than is permitted in the evaluation of F and F'. In particular, the constants A_{2i} in equation (1) should be given with sufficient accuracy. Eleven decimal place numbers are not uncommon here. Second, we have not accounted for machine rounding errors. However, if the computer word length is only one or two digits more than the accuracy required then rounding errors will not interfere significantly with our conclusions. This is because the arithmetic operations required in the algorithm are so few that errors of this kind will not accumulate to a troublesome level. Precise control of rounding errors *can* be realised but only at some computational expense. (See, for example, Rokne and Lancaster (1969)).

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