AN IDEAL PERFORMANCE PROPERTY OF THE MULTICHANNEL WIENER FILTER†

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SUMMARY

Certain combinations of operator length and input length result in multichannel Wiener filters which perform with no error. This property is of interest because it can lead to numerical difficulties if the specified number of filter coefficients exceeds the number required to yield error-free performance.

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Consider a linear discrete multichannel system which consists of k input channels and one output channel. The relation between the input $\underline{x}_t$ and the output y_t is given by the multichannel convolution formula

$$y_t = \sum_{s=o}^{m} \underline{f}_s \, \underline{x}_{t-s} \quad , \quad t = 0,1, \ldots , m+n \qquad (1)$$

where the bar underneath the symbol serves to identify it as a vector (Wiggins and Robinson, 1965). In the present case the filter coefficients $\underline{f}_t$ are k-dimensional row vectors, and the sampled input coefficients $\underline{x}_t$ are k-dimensional column vectors. We assume that the input is of length (n+1),

$$\underline{x}_t = (\underline{x}_o , \underline{x}_1 , \ldots, \underline{x}_n)$$

and that the multichannel filter $\underline{f}_t$ is of length (m+1),

$$\underline{f}_t = (\underline{f}_o , \underline{f}_1 , \ldots, \underline{f}_m)$$

The complete transient convolution (1) can be written as a set of (m+n+1) linear equations,

$$\begin{aligned}
y_o &= \underline{f}_o \, \underline{x}_o \\
y_1 &= \underline{f}_o \, \underline{x}_1 + \underline{f}_1 \, \underline{x}_o \\
&\ldots \\
y_m &= \underline{f}_o \, \underline{x}_m + \underline{f}_1 \, \underline{x}_{m-1} + \ldots + \underline{f}_m \, \underline{x}_o \\
&\ldots \\
y_{m+n} &= \underline{f}_m \, \underline{x}_n
\end{aligned} \qquad (2)$$

The vector products $\underline{f}_s \, \underline{x}_t$ are of the form

$$[f_1(s) \; f_2(s) \; \ldots \; f_k(s)] \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdot \\ \cdot \\ \cdot \\ x_k(t) \end{bmatrix}$$

and hence the sampled output coefficients y_t, t=0,1, . . ., m+n are scalars.

Let us assume that the input $(\underline{x}_o , \underline{x}_1, \ldots, \underline{x}_n)$ and the output $(y_o, y_1, \ldots, y_{m+n})$ are known. Then the system (2) represents (m+n+1) linear simultaneous equations in the (m+1) unknown vector-valued filter coefficients $(\underline{f}_o, \underline{f}_1, \ldots, \underline{f}_m)$. But each vector-valued filter coefficient contains k scalar-valued coefficient elements of the form,

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$$f_i(t)\ ,\ \begin{cases} i = 1,2, \ldots, k \\ t = 0,1, \ldots, m \end{cases}$$

Therefore the system (2) represents (m + n + 1) linear simultaneous equations in the k (m + 1) unknown scalar-valued filter coefficients $f_i(t)$. An ordinary solution of this system, if it exists, can be obtained only when the number of equations is equal to the number of scalar-valued unknown filter coefficients, that is, if

$$(m + n + 1) = k\ (m + 1). \quad (3)$$

It is convenient to let M = m + 1 = filter length and N = n + 1 = input length. We obtain the formula

$$M = \frac{N - 1}{k - 1} \quad (4)$$

which relates the filter length M to the input length N and the number of input channels k when the system (2) has an ordinary solution. Since we deal here with discrete processes, the right member of (4) can only assume integral values, so that

$$M = \left. \frac{N - 1}{k - 1} \right|_{\text{Mod } 1} \quad (5)$$

must be satisfied.

Let us next identify the known output $(y_0, y_1, \ldots, y_{m+n})$ with the known **desired** output $(d_0, d_1, \ldots, d_{m+n})$ that must be specified in order to solve for a k input channel and a single output channel Wiener filter (Wiggins and Robinson, *loc. cit.*). (Equation (1) now becomes

$$d_t = \sum_{s=o}^{m} \underline{f}_s \underline{x}_{t-s} \quad (6)$$

But in view of our previous results, we conclude that the desired output d_t (t = 0,1, . . . , m+n) is obtainable with no error whenever equation (5) is satisfied. Post-multiplying both sides of the **equality** (6) by by $\underline{x}^T_{t-\tau}$ and taking expectations yields

$$E\{d_t \underline{x}^T_{t-\tau}\} = \sum_{s=o}^{m} \underline{f}_s E\{\underline{x}_{t-s} \underline{x}^T_{t-\tau}\},$$

where E is the expectation operator and where the superscript T denotes transposition.

Since $E\{d_t \underline{x}^T_{t-\tau}\} = \underline{g}_\tau$

and $E\{\underline{x}_{t-s} \underline{x}^T_{t-\tau}\} = \underline{r}_{\tau-s}$

we obtain the familiar system of normal equations,

$$\sum_{s=o}^{m} \underline{f}_s \underline{r}_{\tau-s} = \underline{g}_\tau \quad , \tau = 0,1, \ldots, m \quad (7)$$

where $\underline{r}_\tau$ and $\underline{g}_\tau$ are the multichannel input autocorrelation and desired output with input crosscorrelation coefficients, respectively.

It is thus evident that whenever the relation (5) is satisfied the multichannel Wiener filter obtained by solving the system (7) will produce an actual output y_t which is identically equal to the desired output d_t for t = 0,1, . . . , m + n.

The values of N and k determine the filter lengths for which error-free multichannel Wiener filters are obtainable. For the special case k = 1 one has that $M \to \infty$, which is the well known result that in general only infinite length Wiener filters perform with no error.* For k = 2, we have

$$M = N - 1,$$

so that for the 2-channel case exact performance is obtained whenever the input length exceeds the filter length by one sampled value. Table 1 below gives some combinations of operator lengths M and input lengths N that result in error-free Wiener filters for

*Of course exact performance is always achievable in cases for which the compatibility condition $d_t = y_t$ is satisfied (t = 0,1, . . . , m + n). For example, let f_t = (1,2) and x_t = (3,4). Then y_t = (1,2) * (3,4) = (3,10,8). Given x_t = (3,4) and d_t = (3, 10, 8), the Wiener technique yields the filter f_t = (1, 2), and the error is zero.

the k = 3 and k = 6 channel cases. Use of operator lengths for given choices of k and N which are less than those yielding error-free performance leads to non-zero values of this error. Use of operator lengths greater than those leading to perfect performance is to be avoided, since the quest for a number of filter coefficients greater than the number required for this perfect performance can lead to computational difficulties.

k = \ M =	1	2	3 . . .
		N =	
3	3	5	7 . . .
6	6	11	16 . . .

Table 1 — Combinations of filter lengths M and input lengths N which yield error-free Wiener filters for the k = 3 and k = 6 channel cases.

In illustration consider a system for which k = 3. By Table 1 above one notes that perfect performance is achievable for the 2-length filter (M = 2) when the length of the input series $\underline{x}_t$ and of the desired output series $\underline{d}_t$ is N = 5. Thus let

$$\underline{x}_t = \left\{ \begin{bmatrix} -1 \\ 6 \\ 0.5 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 0.6 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 0.7 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 0.8 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ -5 \end{bmatrix} \right\}$$

and

$$d_t = (0.3, 0.5, 1, 0.5, 0.3)$$

The resulting 2-length 3-channel Wiener filter $\underline{f}_t$ can be computed as described by Wiggins and Robinson (loc. cit.), and is

$$\underline{f}_t = \left(\begin{matrix} [-0.10411 & 0.07506 & -0.50900] \\ [-0.54370 & -0.01933 & -0.41177] \end{matrix} \right)$$

When this filter is convolved with the input, one obtains an actual output y_t which is

$$y_t = \underline{f}_t * \underline{x}_t = (0.3, 0.5, 1, 0.5, 0.3, 0.0).$$

As expected, the agreement between d_t and y_t is perfect, and the filter performs with no error.

The present result also holds for multichannel. If there are l such output channels the system (2) generalizes to $l(m) + n$ . . . generalizes to $l(m+n+1)$ equations in $lk(m+1)$ unknowns unknowns, and hence property (5) follows as before. However, the application of most standard lag windows to the autocorrelation coefficients $\underline{r}_\tau$ invalidates our conclusions.

REFERENCE

Wiggins, R. A. and Robinson, E. A., 1965, Recursive solution to the multichannel filtering problem: J. Geophys. Res., 70, 1885-1891.