

THE FINITE FOURIER TRANSFORM

by

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ABSTRACT

The finite Fourier transform of a digitized seismic trace is defined. Many of the elementary properties, which are often simply stated in other papers, are proven. The algorithm, presented by Cooley and Tukey in 1965, which reduces the number of computations required to calculate the finite Fourier transform is developed. This algorithm, called the FFT (Fast Fourier Transform) allows such a fantastic saving in computational time that many of the procedures used to calculate convolutions and correlations of digital values have changed. Circular convolution is defined and is shown to be equivalent to the product of the transforms. The presentation given here does not require a great deal of mathematical sophistication to understand. Although many other transforms — the Fourier Integral transform, the Laplace transform, etc. require a considerable background in mathematics, the finite Fourier transform is simply a manipulation of numbers. For this reason the presentation given here is considered an excellent way of introducing the geophysicist to transform analysis and to deepen his appreciation of processing either in the time domain or equivalently in the frequency domain. Two examples of frequency domain processing, namely, Vibroseis correlation and frequency domain deconvolution, are discussed.

DEFINITION OF THE FINITE FOURIER TRANSFORM

A digitized seismic trace is given by a list or sequence of N numerical time measurements ($x_0, x_1, x_2, \dots, x_{N-1}$). x_0 is the amplitude of the trace at time $t = 0$, x_1 is the amplitude of the trace at time $t = 1\Delta t$, x_2 is the amplitude of the trace at time $t = 2\Delta t$, etc. We have obtained the list or sequence of numbers by taking readings every Δt seconds. Δt is called the sampling interval and is usually equal to .002 seconds. For $\Delta t = .002$ we note the number of samples per second is $1/\Delta t = 500$. From now on we will consider the seismic trace and the list of digitized values to be the same thing, that is, we will refer to the list of numbers as the seismic trace and vice versa. Further, the mathematical term "sequence" will replace "list".

We now define the finite Fourier transform of the seismic trace ($x_0, x_1, x_2, \dots, x_{N-1}$) and develop some of its more elementary properties. Unlike the more sophisticated Fourier transforms, this transform will involve only multiplication and addition of numbers. In particular, using the straight forward approach, $N \times N$ multiply-add operations are required to transform N time measurements to their corresponding frequency values. Later in this article we develop the clever technique which obtains the transform in only $N \times n$ multiply-add operations, where n is much smaller than N . As indicated earlier, a knowledge of the more sophisticated mathematical concepts is not necessary to understand the finite Fourier transform. All that is required is a knowledge of trigonometry and complex numbers. We hope that the basic ideas of transform analysis presented here will increase the readers appreciation of the theoretical and

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practical usefulness of going from the time domain to the frequency domain and back again.

Given the seismic trace $(x_0, x_1, x_2, \dots, x_{N-1})$, we obtain the N spectral values $(X_0, X_1, X_2, \dots, X_{N-1})$ at the discrete frequencies Δf , where $\Delta f = 1/(N\Delta t) = 2f_{NQ}/N$, ($f_{NQ} = 1/(2\Delta t)$) in the following steps:

STEP 1. Multiply $(x_0, x_1, x_2, \dots, x_{N-1})$ with the cosine and negative sine values and distinguish between the two products by using the complex i ($i^2 = -1$).

STEP 2. Add the N products. The result is X_k , the finite Fourier transform at $f = k\Delta f$.

Hence the N frequency values are, by definition,

$$\begin{aligned} X_k = & x_0(\cos 0 - i \sin 0) + x_1(\cos 2\pi k \Delta f 1 \Delta t - i \sin 2\pi k \Delta f 1 \Delta t) \\ & + x_2(\cos 2\pi k \Delta f 2 \Delta t - i \sin 2\pi k \Delta f 2 \Delta t) + \dots \\ & + x_{N-1}(\cos 2\pi k \Delta f (N-1) \Delta t - i \sin 2\pi k \Delta f (N-1) \Delta t). \end{aligned}$$

The angle at which the cosine and sine values are to be evaluated appears quite complicated, but we can immediately simplify since $\Delta f = 1/(N\Delta t)$ or $\Delta f \Delta t = 1/N$.

Making this substitution of $1/N$ for $\Delta f \Delta t$ and using the summation notation we write the finite Fourier transform as

$$X_k = \sum_{n=0}^{N-1} x_n \left[\cos \frac{2\pi k n}{N} - i \sin \frac{2\pi k n}{N} \right] \quad k = 0, 1, 2, \dots, N-1. \quad (1)$$

Summation notation

Without the use of the summation notation the equation for X_k is quite unwieldy, as seen below.

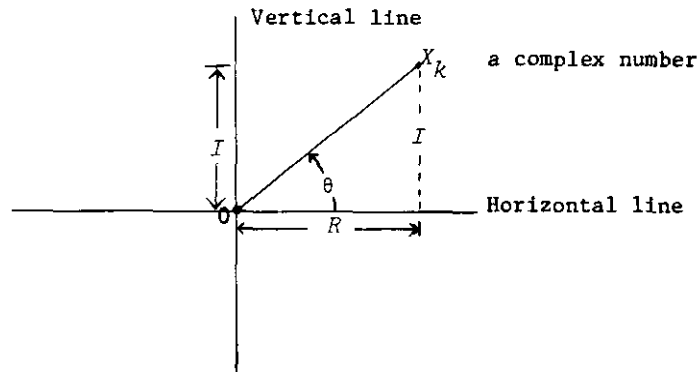
$$\begin{aligned} X_k = & x_0(\cos 0 - i \sin 0) + x_1 \left[\cos \frac{2\pi k 1}{N} - i \sin \frac{2\pi k 1}{N} \right] \\ & + x_2 \left[\cos \frac{2\pi k 2}{N} - i \sin \frac{2\pi k 2}{N} \right] + \dots + x_{N-1} \left[\cos \frac{2\pi k (N-1)}{N} - i \sin \frac{2\pi k (N-1)}{N} \right] \\ & \text{for } k = 0, 1, 2, \dots, N-1. \end{aligned}$$

That is, it is only for compactness that we use the summation notation, as given in equation (1), to define the finite Fourier transform.

As can be seen by equation (1) (or its expanded version) each frequency value X_k requires N complex multiply-add operations between the time measurements and the trigonometric values. Therefore, to completely

obtain the N frequency measurements requires $N \times N$ multiply-add operations. We observe from equation (1) that the finite Fourier transform of a finite list of numbers $(x_0, x_1, x_2, \dots, x_{N-1})$ (time values) is another finite list of complex numbers $(X_0, X_1, X_2, \dots, X_{N-1})$ (frequency values). The frequency values constitutes a frequency analysis of the seismic trace at discrete frequencies $\Delta f = 2f_{NQ}/N$. This set of complex numbers $(X_0, X_1, X_2, \dots, X_{N-1})$ is the finite Fourier transform. The set of real numbers $(A_0, A_1, A_2, \dots, A_{N-1})$, where $A_k = \text{length of } X_k$, is defined to be the amplitude spectrum of the seismic trace. The other set of real numbers (also obtained from X_k) $(\theta_0, \theta_1, \theta_2, \dots, \theta_{N-1})$, where $\theta_k = \text{angle of } X_k$, is defined to be the phase spectrum of the seismic trace.

The length and angle of a complex number is best illustrated using the complex plane. As we all know a complex number is a pair of real numbers. Hence it can be represented in the plane using a horizontal line (real part) and a vertical line (imaginary part) as follows:



The length of X_k is the length of the line from 0 to X_k . By the Pythagorean theorem this length is $\sqrt{R^2 + I^2}$. The length of X_k is denoted by $|X_k|$ which can be read as "the length of X_k ". The phase of X_k is the angle the length line makes with the horizontal axes. If we let θ denote the phase of X_k then from the diagram $\tan \theta = I/R$. To denote the real and imaginary parts of X_k we often write $X_k = R + iI$.

To restate, the Finite Fourier Transform of a seismic trace is a list of complex numbers $(X_0, X_1, X_2, \dots, X_{N-1})$ and the amplitude spectrum is the length of X_k and the phase spectrum is the angle of X_k .

We now proceed to write equation (1) in a different way. The new equations for X_k will make the proofs of the elementary properties of the finite Fourier transform much simpler.

Anyone familiar with complex numbers will recognize the term $W = \cos(2\pi/N) - i \sin(2\pi/N)$ as one of the N^{th} roots of 1, that is $W^N = 1$. There are N distinct roots of 1 given by $\cos(2\pi m/N) - i \sin(2\pi m/N)$ for $m = 0, 1, 2, \dots, N-1$. It is well known (De Moirre's Theorem) that

$$\left(\cos \frac{2\pi}{N} - i \sin \frac{2\pi}{N} \right)^{kn} = \cos \frac{2\pi kn}{N} - i \sin \frac{2\pi kn}{N}.$$

Using this relation and letting $W = \cos(2\pi/N) - i \sin(2\pi/N)$ we rewrite equation (1) as

$$X_k = \sum_{n=0}^{N-1} x_n W^{kn} \quad \text{for } k = 0, 1, 2, \dots, N-1. \quad (2)$$

It follows from equation (2) that given the N time values $(x_0, x_1, x_2, \dots, x_{N-1})$ and the N^{th} roots of unity we can obtain the transform by forming products and adding.

We can also rewrite equation (2) by using the identity

$$e^{-i \frac{2\pi}{N}} = \cos \frac{2\pi}{N} - i \sin \frac{2\pi}{N} \quad \text{to obtain}$$

$$X_k = \sum_{n=0}^{N-1} x_n e^{-i \frac{2\pi kn}{N}} \quad \text{for } k = 0, 1, 2, \dots, N-1 \quad (3)$$

This is the form used by most authors to define the finite Fourier transform.

All the laws governing the taking of a number to a power are valid in equations (2) and (3). That is, we have

$$e^0 = 1 \quad \text{or } W^0 = 1$$

$$e^a e^b = e^{a+b} \quad \text{or } W^a W^b = W^{a+b}$$

$$e^a e^{-b} = e^{a-b} \quad \text{or } W^a W^{-b} = W^{a-b} \quad \text{Note: } e^{-a} = 1/e^a$$

$$(e^a)^n = e^{na} \quad \text{or } (W^a)^n = W^{na} \quad (a \text{ positive or negative})$$

The power laws make many of the properties of the finite Fourier transform obvious.

We now take up some special spectral values. The zeroth spectral value is

$$X_0 = \sum_{n=0}^{N-1} x_n W^{0n} \quad k = 0. \quad \text{But } W^0 = 1, \text{ hence}$$

$$X_0 = \sum_{n=0}^{N-1} x_n = x_0 + x_1 + x_2 + \dots + x_{N-1}.$$

In other words, we have demonstrated that the zeroth spectral value is simply the sum of all the values of the input seismic trace. There are N spectral values, and another very interesting one is $X_{N/2}$, $k=N/2$ which is the spectral value at

$$f = (N/2) \Delta f = (N/2) (2f_{NQ}/N) = f_{NQ} \text{ cps, the Nyquist frequency.}$$

$$\text{Now} \quad W^{N/2} = \left(\cos \frac{2\pi}{N} - i \sin \frac{2\pi}{N} \right)^{N/2} = \cos \pi - i \sin \pi = -1.$$

$$\text{Hence} \quad X_{N/2} = \sum_{n=0}^{N-1} x_n W^{(N/2)n} \quad k = N/2. \quad \text{But } W^{N/2} = -1.$$

$$\text{Hence} \quad X_{N/2} = \sum_{n=0}^{N-1} x_n (-1)^n = x_0 - x_1 + x_2 - x_3 + \dots - x_{N-1}.$$

We have demonstrated that the spectral value at the Nyquist frequency is simply the plus and minus of all the input values.

We now discuss some of the properties of the finite Fourier transform.

PROPERTY 1

Equation (2) goes from the time domain to the frequency domain. A very similar equation defining the inverse finite Fourier transform, transforms the discrete spectral values back to the original discrete time values. That is, given the spectral values $(X_0, X_1, X_2, \dots, X_{N-1})$ we obtain the original time values $(x_0, x_1, x_2, \dots, x_{N-1})$ from

$$x_n = \frac{1}{N} \sum_{k=0}^{N-1} X_k W^{-kn} \quad \text{for } n = 0, 1, 2, \dots, N-1. \quad (4)$$

To show that equation (4) is valid we proceed as follows:

First, we substitute for X_k its value as given by equation (2). Doing this equation (4) becomes

$$\frac{1}{N} \sum_{k=0}^{N-1} \left(\sum_{r=0}^{N-1} x_r W^{kr} \right) W^{-kn} = \frac{1}{N} \sum_{k=0}^{N-1} \left(\sum_{r=0}^{N-1} x_r W^{k(r-n)} \right).$$

Interchanging the order of summation we obtain

$$\frac{1}{N} \sum_{r=0}^{N-1} \left(\sum_{k=0}^{N-1} x_r W^{k(r-n)} \right).$$

What we wish to show is that this double summation equals the original time value x_n . It should be clear that this double summation will be equal to x_n if the term in the brackets is 0 for all r except $r = n$ and for $r = n$ the term in the brackets is Nx_n . Obviously, for $r = n$ the term in the brackets is

$$\sum_{k=0}^{N-1} x_n W^{k(n-n)} = \sum_{k=0}^{N-1} x_n W^0 = Nx_n.$$

For $r \neq n$, we have to show it is zero. Let $p = r - n$, then

$$\sum_{k=0}^{N-1} x_r W^{kp} = x_r \sum_{k=0}^{N-1} (W^p)^k \quad \text{and} \quad \sum_{k=0}^{N-1} (W^p)^k$$

is a geometric series $1 + W^p + (W^p)^2 + (W^p)^3 + \dots + (W^p)^{N-1}$ whose common ratio $r = W^p$. Now we all know that $1 + r + r^2 + \dots + r^{N-1} = \frac{1-r^N}{1-r}$.

Hence

$$\sum_{k=0}^{N-1} (W^p)^k = 1 + W^p + (W^p)^2 + \dots + (W^p)^{N-1} = \frac{1-W^{pN}}{1-W^p} = \frac{1-(W^N)^p}{1-W^p}$$

But W is an N^{th} root of 1 or $W^N = 1$ and it follows that the numerator is zero. From this it follows that the equation (4) is valid, that is,

$$x_n = \frac{1}{N} \sum_{k=0}^{N-1} X_k W^{-kn}.$$

In other words equation (4), which defines the inverse transform, gives the original time values from the spectral values.

Here we see that the inverse transform is obtained simply by taking the product of the spectral values and appropriate N^{th} roots of unity and summing.

PROPERTY 2

From the definitions of the transform and the inverse transform, the seismic trace and its spectral values are periodic with period N . To be sure, the digitally recorded seismic trace $(x_0, x_1, x_2, \dots, x_{N-1})$ is finite and not periodic but we shall see that the values, as defined by equation (4), are periodic, that is, $x_{n+N} = x_n$ and therefore, infinitely long. To see that this is the case we proceed as follows.

$$\begin{aligned}
 x_n &= \frac{1}{N} \sum_{k=0}^{N-1} X_k W^{-kn} \\
 x_{n+N} &= \frac{1}{N} \sum_{k=0}^{N-1} X_k W^{-k(n+N)} = \frac{1}{N} \sum_{k=0}^{N-1} X_k W^{-kn} W^{-kN} \\
 &= \frac{1}{N} \sum_{k=0}^{N-1} X_k W^{-kn} (W^N)^{-k}
 \end{aligned}$$

but W is an N^{th} root of 1 or $W^N = 1$, hence $(W^N)^{-k} = (1)^{-k} = 1$. It follows that

$$x_{n+N} = \frac{1}{N} \sum_{k=0}^{N-1} X_k W^{-kn} = x_n$$

as we wished to show. Similarly, the frequency values are periodic with period $2f_{NQ}$, that is, $X_{k+N} = X_k$. This proves property (2) which states that both the time domain and frequency domain representation as given by the finite Fourier transform are periodic. Therefore, for mathematical convenience and for easier comprehension of the kind of convolution and correlation that is obtained from the product of finite Fourier transforms the infinite periodic lists, namely,

	Principal part	repeats
$\dots, x_0, \dots, x_{N-1},$	$x_0, x_1, x_2, \dots, x_{N-1},$	$x_0,$
$\dots, X_0, \dots, X_{N-1},$	$X_0, X_1, X_2, \dots, X_{N-1},$	$X_0,$
INDEX $-N, \dots, -1,$	$0, 1, 2, \dots, N-1,$	N

will be considered to be a Fourier pair.

PROPERTY 3

The finite Fourier transform is a linear operation. That is, given two seismic traces $(x_0, x_1, x_2, \dots, x_{N-1})$ and $(y_0, y_1, y_2, \dots, y_{N-1})$ the finite Fourier transform of the stack of these two traces is the stack of the individual transforms.

The proof of this follows from equation (2) which defines the transform of the stack to be

$$\begin{aligned}
 S_k &= \sum_{n=0}^{N-1} (x_n + y_n) W^{kn} = \sum_{n=0}^{N-1} (x_n W^{kn} + y_n W^{kn}) \\
 &= \sum_{n=0}^{N-1} x_n W^{kn} + \sum_{n=0}^{N-1} y_n W^{kn} \\
 &= X_k + Y_k.
 \end{aligned}$$

This shows that the transform of the stack is the stack or sum of the individual transforms.

In general, if a and b are any constants then the finite Fourier transform of $s_n = ax_n + by_n$ is given by $S_k = aX_k + bY_k$.

Although we have implied the time values to be real, the finite Fourier transform and its inverse have the same properties when applied to complex lists. For example, if $(z_0, z_1, z_2, \dots, z_{N-1})$ is a sequence of complex numbers, then its transform is

$$Z_k = \sum_{n=0}^{N-1} z_n W^{kn}.$$

The original time values can also be obtained by the inverse transform of Z_k , that is,

$$z_n = \frac{1}{N} \sum_{k=0}^{N-1} Z_k W^{-kn}.$$

Although the seismic trace is always a sequence of real numbers we can consider two seismic traces $(x_0, x_1, x_2, \dots, x_{N-1})$ and $(y_0, y_1, y_2, \dots, y_{N-1})$ to be one complex sequence $(z_0, z_1, z_2, \dots, z_{N-1})$, where $z_n = x_n + iy_n$. After we have discussed property 4, a technique of calculating the transform of two seismic traces by doing only one finite Fourier Transform will be given. This, of course, cuts the computational time of any finite Fourier transform algorithm by one-half.

PROPERTY 4

If $(x_0, x_1, x_2, \dots, x_{N-1})$ is real, then the second part of the finite Fourier transform is equal to the complex conjugate of the first part. In other words given the spectral values

$$(X_0, X_1, X_2, \dots, X_{N-1}),$$

the first and second parts are as follows:

$$\underbrace{X_0, X_1, X_2, \dots, X_{(N/2)-1}}_{\text{First Part}}, \underbrace{X_{N/2}, X_{(N/2)+1}, \dots, X_{N-2}, X_{N-1}}_{\text{Second Part}}.$$

What we wish to show is that

$$\begin{aligned} X_{N-1} &= X_1^* && \text{*indicates complex conjugate} \\ X_{N-2} &= X_2^* \\ &\vdots && \vdots \\ &\vdots && \vdots \\ X_{(N/2)+1} &= X_{(N/2)-1}^* \end{aligned}$$

In general, we want to show that

$$X_{N-j} = X_j^*.$$

To do this we observe that by equation (2)

$$X_{N-j} = \sum_{n=0}^{N-1} x_n W^{(N-j)n} \quad k = N - j$$

$$X_{N-j} = \sum_{n=0}^{N-1} x_n W^{Nn} W^{-jn} \quad \text{now } W^{Nn} = 1$$

as W is an N^{th} root of 1, that is, $W^N = 1$. Hence $W^{Nn} = (W^N)^n = 1$. So we have

$$X_{N-j} = \sum_{n=0}^{N-1} x_n W^{-jn}.$$

Now $W^{-jn} = (W^{-1})^{jn}$ and $W = \cos \frac{2\pi}{N} - i \sin \frac{2\pi}{N}$

$$W^{-1} = \frac{1}{W} = \frac{1}{\cos \frac{2\pi}{N} - i \sin \frac{2\pi}{N}}.$$

We now multiply top and bottom by $\cos(2\pi/N) + i \sin(2\pi/N)$ to obtain

$$W^{-1} = \frac{1}{\left[\cos \frac{2\pi}{N} - i \sin \frac{2\pi}{N} \right]} \frac{\cos \frac{2\pi}{N} + i \sin \frac{2\pi}{N}}{\cos \frac{2\pi}{N} + i \sin \frac{2\pi}{N}}$$

or $W^{-1} = \frac{\cos \frac{2\pi}{N} + i \sin \frac{2\pi}{N}}{\cos^2 \left(\frac{2\pi}{N} \right) + \sin^2 \left(\frac{2\pi}{N} \right)} = \frac{\cos \frac{2\pi}{N} + i \sin \frac{2\pi}{N}}{1}$

or $W^{-1} = \left[\cos \frac{2\pi}{N} - i \sin \frac{2\pi}{N} \right]^* = (W)^*.$

Hence $W^{-jn} = (W^{-1})^{jn} = (W^*)^{jn} = (W^{jn})^*.$

Since x_n is real, we have $x_n^* = x_n$.

It follows that

$$X_{N-j} = \sum_{n=0}^{N-1} x_n w^{-jn} = \sum_{n=0}^{N-1} x_n^* (w^{jn})^* = \left(\sum_{n=0}^{N-1} x_n w^{jn} \right)^* = (X_j)^*$$

or $X_{N-j} = (X_j)^*$ as we wished to show.

In other words, given N real input values the transform has 2 real values X_0 and $X_{N/2}$ and only $N/2 - 1$ independent complex values. Now one complex number consists of two real numbers. Hence, the equivalent number of real numbers in the transform is $2 + 2(N/2 - 1) = N$, which is as it should be.

PROPERTY 5

Using property 3 and 4 we will now show how to obtain the finite Fourier transform of two seismic traces for the price of one.

Let $(x_0, x_1, x_2, \dots, x_{N-1})$ and $(y_0, y_1, y_2, \dots, y_{N-1})$ be the two seismic traces. We form one complex sequence $(z_0, z_1, z_2, \dots, z_{N-1})$ where $z_n = x_n + iy_n$. The finite Fourier transform of this one complex sequence is calculated. The result is $(Z_0, Z_1, Z_2, \dots, Z_{N-1})$. We now derive the equations which give the spectral values X_k and Y_k of the two seismic traces from Z_k . This in effect calculates the transform of two seismic traces for the price of one as most transform algorithms have complex valued inputs.

Since z_n is the stack of x_n and iy_n it follows from Property 3 that

$$z_k = X_k + iY_k \quad \text{for } k = 0, 1, 2, 3, \dots, N-1.$$

Hence, it also follows that $z_{N-k} = X_{N-k} + iY_{N-k}$.

From property 4, we have that

$$X_{N-k} = X_k^* \quad \text{and} \quad Y_{N-k} = Y_k^*.$$

Let $X_k = a + ib$ and $Y_k = c + id$, then it follows from

$$\text{and} \quad \begin{aligned} z_k &= X_k + iY_k \\ z_{N-k} &= X_k^* + iY_k^* \end{aligned}$$

$$\text{that} \quad z_k = a + ib + i(c + id) = (a - d) + i(b + c)$$

$$z_{N-k} = a - ib + i(c - id) = (a + d) - i(b - c).$$

Hence it follows that

$$z_{N-k}^* = (a + d) + i(b - c).$$

It should be remembered that the complex conjugate is obtained by changing all i to $-i$.

$$\text{We now have } Z_k = (a - d) + i(b + c)$$

$$\text{and } Z_{N-k}^* = (a + d) + i(b - c).$$

It follows immediately that $(Z_k + Z_{N-k}^*)/2 = a + ib = X_k$

$$\text{and } (Z_k - Z_{N-k}^*)/2i = c + id = Y_k.$$

As can be seen, the above equations give the spectral values of both seismic traces from the spectral values Z_k which is the finite Fourier transform of $z_n = x_n + iy_n$.

DEFINITION OF PERIODIC OR CIRCULAR CORRELATION AND CONVOLUTION

Given two seismic traces and their periodic representation the circular correlation is defined by

$$C_{xy}'(n) = \sum_{t=0}^{N-1} x_{t+n} y_t \quad \text{for } n = 0, 1, 2, \dots, N-1. \quad (5)$$

It is to be noted that x_{N+n} is defined by $x_{N+n} = x_n$. It therefore follows that after x_{N-1} we do not have zeros but a repetition of the previous N values. The circular correlation of x_n and y_n recognizes this periodic repetition and for this reason is not equal to the ordinary correlation. For example,

$$C_{xy}'(1) = (x_1 y_0 + x_2 y_1 + \dots + x_{N-1} y_{N-2} + x_0 y_{N-1})$$

while the ordinary correlation is

$$(x_1 y_0 + x_2 y_1 + \dots + x_{N-1} y_{N-2} + 0).$$

The difference in the last term is due to the values of x_n repeating after x_{N-1} . In property 6 we show that the straight forward product of the finite Fourier transforms is equivalent to circular correlation, not ordinary correlation.

Given a seismic trace $(x_0, x_1, x_2, \dots, x_{N-1})$ and a filter $(y_0, y_1, y_2, \dots, y_{N-1})$ the circular convolution is defined by

$$C_{xy}(n) = \sum_{t=0}^{N-1} y_t x_{n-t} \quad \text{for } n = 0, 1, 2, \dots, N-1. \quad (6)$$

Here again the circular convolution recognizes the periodic representation of the sequences. For example,

$$C_{xy}(1) = y_0 x_1 + y_1 x_0 + y_2 x_{N-1} + \dots + y_{N-1} x_2$$

whereas the ordinary convolution at $n = 1$ is simply

$$y_0 x_1 + y_1 x_0 + 0 + 0 + \dots + 0.$$

It should be noted that for either correlation or convolution the two sequences will be of equal length. This is always possible as the shorter one can be extended with zeros to equal the other.

PROPERTY 6

We shall now show that convolution and correlation can be performed in the frequency domain by forming appropriate products of the spectral values. We first consider correlation.

Given the spectral values X_k and Y_k , we show that the product $X_k Y_k^*$ is the frequency domain representation of the circular correlation sequence $C'_{xy}(n)$ for $n = 0, 1, 2, \dots, N-1$. To prove this most important property we could show either that the direct finite Fourier transform of $C'_{xy}(n)$ is $X_k Y_k^*$, or show that the inverse transform of $X_k Y_k^*$ is $C'_{xy}(n)$. We choose to do the first.

The finite Fourier transform of $C'_{xy}(n)$ is

$$\sum_{n=0}^{N-1} C'_{xy}(n) W^{kn}$$

but $C'_{xy}(n) = \sum_{t=0}^{N-1} x_{t+n} y_t$. We substitute this in the above, thus obtaining

$$= \sum_{n=0}^{N-1} \left(\sum_{t=0}^{N-1} x_{t+n} y_t \right) W^{kn}.$$

We now interchange the order of summation to obtain

$$= \sum_{t=0}^{N-1} y_t \left(\sum_{n=0}^{N-1} x_{t+n} W^{kn} \right).$$

Next, we note that $W^{kt} W^{-kt} = 1$ and hence we rewrite the above to give

$$= \sum_{t=0}^{N-1} y_t \left(\sum_{n=0}^{N-1} x_{t+n} W^{k(t+n)} \right) W^{-kt}.$$

Now, for all values of t , the term in the brackets is X_k . This follows since x_n and W are periodic with period N .

Substituting in X_k , the above expression becomes

$$= X_k \sum_{t=0}^{N-1} y_t W^{-kt}$$

or $= X_k Y_k^*$ as $Y_k^* = \sum_{t=0}^{N-1} y_t W^{-kt}$ when y_t is real.

In other words, we have shown that the finite Fourier transform of $C'_{xy}(n)$ is $X_k Y_k^*$, that is

$$X_k Y_k^* = \sum_{n=0}^{N-1} C'_{xy}(n) W^{kn}.$$

To restate, the product $X_k Y_k^*$ for $k = 0, 1, 2, \dots, N-1$ is the frequency domain representation of the circular correlation of x_n and y_n . It should now be clear how to obtain the correlation sequence $C'_{xy}(n)$ of x_n and

y_n without performing the operations $\sum_{t=0}^{N-1} x_{t+n} y_t$ for $n = 0, 1, 2, \dots, N-1$

The procedure is, first, calculate the finite Fourier transform of x_n and y_n giving X_k and Y_k , second, form the products $X_k Y_k^*$, and finally calculate the inverse transform of this product. The result is the complete set of N -lag values of $C'_{xy}(n)$. Of course, the circular correlation is of no interest to anyone. What we desire is the ordinary correlation coefficients of x_n and y_n . We shall now try to make it obvious that by adding N zeros to the ends of x_n and y_n the circular correlation of the zero filled periodic sequences (period = $2N$) is the ordinary correlation of the original sequences.

The zero filled periodic sequences are

$$x_0, x_1, x_2, \dots, x_{N-1}, 0, 0, 0, \dots, 0, x_0, x_1, \dots$$

+ repeats repeats +

$$y_0, y_1, y_2, \dots, y_{N-1}, 0, 0, 0, \dots, 0, y_0, y_1, \dots$$

Now the first lag value of these double length sequences is obtained by shifting y_n one unit to the right, forming products and adding. The result is $C'_{xy}(1) = x_1 y_0 + x_2 y_1 + x_3 y_2 + \dots + x_{N-1} y_{N-2} + 0$.

This of course, is the ordinary correlation of the original sequences. Similarly, it can be shown that the other lag values

$$C'_{xy}(n) \text{ for } n = 0, 1, 2, \dots, N-1$$

are the ordinary correlation coefficients for all positive lags.

Further, it should be clear that $C'_{xy}(N) = 0$, that $C'_{xy}(N+1)$ is the correlation at a lag of $-N+1$, and that $C'_{xy}(2N-1)$ is the correlation at a lag of -1 .

Hence, it follows that the circular correlation of the zero filled sequences contains all the positive and negative lagged correlation values of x_n and y_n .

Therefore, the procedure to obtain the correlation function of two seismic traces $(x_0, x_1, x_2, \dots, x_{N-1})$ and $(y_0, y_1, y_2, \dots, y_{N-1})$ is as follows:

1. Add N zeros to the end of each sequence.
2. Calculate the finite Fourier transform of these extended sequences to give X_k and Y_k .
3. Form the product $X_k Y_k^*$ for $k = 0, 1, 2, \dots, N, \dots, 2N-1$.
4. Calculate the inverse transform of these products.

The result is the $2N$ correlation coefficients of the seismic traces. The above procedure is referred to as the frequency domain calculation of the correlation values. Further, if the Fast Fourier Transform, to be discussed shortly, is used in steps 2 and 4, then the above method is called "fast correlation".

PROPERTY 7

Previously, we discussed the correlation of two sequences and gave the equivalent frequency domain operation. We now discuss the circular convolution of two sequences.

It will be shown that if X_k and Y_k are the transforms of x_n and y_n , then the frequency domain representation of the circular convolution

$$C_{xy}(n) = \sum_{t=0}^{N-1} y_t x_{n-t}$$

is given by the product $X_k Y_k$. To prove this we show that the finite Fourier transform of $C_{xy}(n)$ is $X_k Y_k$.

The finite Fourier transform of $C_{xy}(n)$ is

$$\begin{aligned} & \sum_{n=0}^{N-1} C_{xy}(n) W^{kn} \\ &= \sum_{n=0}^{N-1} \left(\sum_{t=0}^{N-1} y_t x_{n-t} \right) W^{kn}. \end{aligned}$$

We interchange the order of summation and note that $W^{kt} W^{-kt} = 1$, thus obtaining

$$= \sum_{t=0}^{N-1} y_t \left(\sum_{n=0}^{N-1} x_{n-t} W^{k(n-t)} \right) W^{kt}.$$

The term in the bracket is X_k as x_n and W are periodic.

Hence,

$$\begin{aligned} &= X_k \sum_{t=0}^{N-1} y_t W^{kt} \\ &= X_k Y_k \end{aligned}$$

as we wished to show.

It follows immediately that the circular convolution may be calculated in the frequency domain. The procedure is, first calculate the finite Fourier transform of x_n and y_n giving X_k and Y_k , second form the product $X_k Y_k$, and finally calculate the inverse transform of this product. Of course, circular convolution is not ordinary convolution, that is, it is not filtering. It should be clear that if the filter is of length M and the trace is of length N , then the circular convolution of the trace extended by M zeros and the filter extended by N zeros is the ordinary convolution of the original sequences.

Therefore the procedure to obtain the convolution of $(x_0, x_1, x_2, \dots, x_{N-1})$ and $(y_0, y_1, \dots, y_{M-1})$ is as follows:

1. Add M zeros to x_n and N zeros to y_m .
2. Calculate the finite Fourier transform of these extended sequences to give X_k and Y_k .
3. Form the product $X_k Y_k$ for $k = 0, 1, 2, \dots, M, \dots, N, \dots, M+N-1$.
4. Calculate the inverse transform of these products.

The result is the $M+N$ convolved output. The above procedure is referred to as performing convolution in the frequency domain. Further, if the Fast Fourier Transform algorithm is used in steps 2 and 4, then the above method is called "fast convolution". We now describe the clever technique discovered by Cooley and Tukey in 1965 which considerably reduces the computer time required to calculate the finite Fourier transform.

THE FAST FOURIER TRANSFORM FFT

The finite Fourier transform of x_n , as defined in the previous section, is given by the equation

$$X_k = \sum_{n=0}^{N-1} x_n W^{kn}, \quad \text{where } W = \cos(2\pi/N) - i \sin(2\pi/N).$$

If we do away with the summation symbol the above equation becomes

$$X_k = x_0 + x_1 W^k + x_2 W^{2k} + \dots + x_{N-1} W^{(N-1)k} \quad \text{for } k = 0, 1, 2, \dots, N-1. \quad (7)$$

It was shown earlier that X_k is the spectral value at the frequency $f = k(2f_{NQ}/N)$. This can also be considered to be the k^{th} harmonic frequency of the periodic sequence $(x_0, x_1, x_2, \dots, x_{N-1})$. To see this we note that N periodic values with a sampling interval of Δt has a fundamental period of $N\Delta t$ and hence the fundamental harmonic is $f = \frac{1}{N\Delta t}$ cps

But $\frac{1}{N\Delta t} = 2f_{NQ}/N$ and hence $k(2f_{NQ}/N)$ can be considered to be the k^{th} harmonic frequency of the periodic sequence.

To become more familiar with operations of equation (7) we now take up an example. This example will also be used to describe the Fast Fourier Transform technique for $N = 8$.

Suppose the seismic trace is $(x_0, x_1, x_2, \dots, x_{N-1})$, then the finite Fourier transform as given by (7) is the following

$$X_0 = x_0 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$$

$$X_1 = x_0 + x_1W + x_2W^2 + x_3W^3 + x_4W^4 + x_5W^5 + x_6W^6 + x_7W^7$$

$$X_2 = x_0 + x_1W^2 + x_2W^4 + x_3W^6 + x_4 + x_5W^2 + x_6W^4 + x_7W^6$$

$$X_3 = x_0 + x_1W^3 + x_2W^6 + x_3W^1 + x_4W^4 + x_5W^7 + x_6W^2 + x_7W^5$$

$$X_4 = x_0 + x_1W^4 + x_2 + x_3W^4 + x_4 + x_5W^4 + x_6 + x_7W^4$$

$$X_5 = x_0 + x_1W^5 + x_2W^2 + x_3W^7 + x_4W^4 + x_5W^1 + x_6W^6 + x_7W^3$$

$$X_6 = x_0 + x_1W^6 + x_2W^4 + x_3W^2 + x_4 + x_5W^6 + x_6W^4 + x_7W^2$$

$$X_7 = x_0 + x_1W^7 + x_2W^6 + x_3W^5 + x_4W^4 + x_5W^3 + x_6W^2 + x_7W^1$$

It should be noted that $W = \cos \frac{2\pi}{8} - i \sin \frac{2\pi}{8}$ in the above equations. This was used to simplify the above equations as $W^{8+n} = W^n$. Further, $W^4 = \left[\cos \frac{2\pi}{8} - i \sin \frac{2\pi}{8} \right]^4 = \cos 4 \left(\frac{2\pi}{8} \right) - i \sin 4 \left(\frac{2\pi}{8} \right) = -1$. It should be clear that each equation involves 8 multiply-add operations and there are 8 equations. It follows that $8 \times 8 = 64$ multiply-add operations are required to calculate the transform assuming the straight forward approach is used. For a seismic trace of length 1000 the straight forward approach would require 1,000,000 multiply-add operations. In general the computation of the finite Fourier transform of N data values requires N^2 multiply-add

operations. For N quite large and having more than one trace to transform this number of multiply-add operations requires considerable computer time. It was for this reason attempts were made to reduce the number of multiply-add operations and in 1965 Cooley and Tukey reported a method to do this. It is called the Fast Fourier Transform or FFT. Using this method the number of multiply-add operations for a seismic trace of length 1000 is reduced from 1,000,000 to only 10,000 operations. We now develop the FFT method for the case of $N = 8$.

Briefly, the method consists of dividing the data values in half to give 2 seismic traces each of length 4 and then dividing these in half to give 4 seismic traces each of length 2. It will be shown that the finite Fourier transform of numerous short sequences requires fewer operations than the transform of one long sequence. This is the basis of the FFT method.

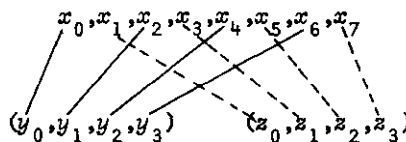
It should be noted that the finite Fourier transform of one data value x_0 is simply $X_0 = x_0$. That is, the frequency measurement and the time measurement for one data value are the same. The finite Fourier transform for two data values (x_0, x_1) , $N = 2$ is also very easy to calculate. Using equation (7), with $W = \cos \frac{2\pi}{2} - i \sin \frac{2\pi}{2} = \cos \pi - i \sin \pi = -1$,

we obtain

$$X_0 = x_0 + x_1$$

$$X_1 = x_0 + x_1 W = x_0 - x_1.$$

Now for the case of $N = 8$, the input is $(x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ and we divide this sequence in half to obtain two sequences as follows:



The y sequence consists of all the original data values with even subscripts, that is $y_0 = x_0, y_1 = x_2, etc.$ The z sequence consists of all the original data values with odd subscripts. Both sequences (y_0, y_1, y_2, y_3) and (z_0, z_1, z_2, z_3) have $N = 4$. It follows that the finite Fourier transform of (y_0, y_1, y_2, y_3) is

$$Y_k = \sum_{n=0}^3 y_n \left(\cos \frac{2\pi}{4} - i \sin \frac{2\pi}{4} \right)^{kn} \quad \text{for } k = 0, 1, 2, 3.$$

Using De Moivre's Theorem this can be written as

$$Y_k = \sum_{n=0}^3 y_n \left(\cos \frac{2\pi}{8} - i \sin \frac{2\pi}{8} \right)^{2kn} \quad \text{for } k = 0, 1, 2, 3.$$

or

$$Y_k = \sum_{n=0}^3 y_n W^{2kn} \quad \text{for } k = 0, 1, 2, 3.$$

Also

$$Z_k = \sum_{n=0}^3 z_n W^{2kn}. \quad \text{for } k = 0, 1, 2, 3.$$

Expanding these equations in W we obtain the finite Fourier transforms of (y_0, y_1, y_2, y_3) and (z_0, z_1, z_2, z_3) to be

$$k = 0 \quad Y_0 = y_0 + y_1 + y_2 + y_3 = x_0 + x_2 + x_4 + x_6$$

$$k = 1 \quad Y_1 = y_0 + y_1 W^2 + y_2 W^4 + y_3 W^6$$

$$k = 2 \quad Y_2 = y_0 + y_1 W^4 + y_2 + y_3 W^4$$

$$k = 3 \quad Y_3 = y_0 + y_1 W^6 + y_2 W^4 + y_3 W^2$$

$$k = 0 \quad Z_0 = z_0 + z_1 + z_2 + z_3 = x_1 + x_3 + x_5 + x_7$$

$$k = 1 \quad Z_1 = z_0 + z_1 W^2 + z_2 W^4 + z_3 W^6$$

$$k = 2 \quad Z_2 = z_0 + z_1 W^4 + z_2 + z_3 W^4$$

$$k = 3 \quad Z_3 = z_0 + z_1 W^6 + z_2 W^4 + z_3 W^2.$$

To illustrate the first step in the development of the FFT we now develop the equations which give the spectral values $(X_0, X_1, X_2, X_3, X_4, X_5, X_6, X_7)$ in terms of the spectral values of the two shorter sequences.

We know from equation (7) that

$$X_k = \sum_{n=0}^7 x_n W^{kn} \quad \text{for } k = 0, 1, 2, 3, 4, 5, 6, 7.$$

Expanding the above equation for X_0 we obtain

$$X_0 = x_0 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7.$$

On simply regrouping the terms this can be written as

$$X_0 = (x_0 + x_2 + x_4 + x_6) + (x_1 + x_3 + x_5 + x_7) = Y_0 + Z_0.$$

Also $X_1 = x_0 + x_1 W + x_2 W^2 + x_3 W^3 + x_4 W^4 + x_5 W^5 + x_6 W^6 + x_7 W^7$ or

$$X_1 = (x_0 + x_2 W^2 + x_4 W^4 + x_6 W^6) + W(x_1 + x_3 W^2 + x_5 W^4 + x_7 W^6) = Y_1 + WZ_1.$$

$$\text{Also } X_2 = x_0 + x_1 W^2 + x_2 W^4 + x_3 W^6 + x_4 + x_5 W^2 + x_6 W^4 + x_7 W^6 \quad \text{or}$$

$$X_2 = (x_0 + x_2 W^4 + x_4 + x_6 W^4) + W^2(x_1 + x_3 W^4 + x_5 + x_7 W^4) = Y_2 + W^2 Z_2.$$

$$\text{Also } X_3 = x_0 + x_1 W^3 + x_2 W^6 + x_3 W^9 + x_4 W^4 + x_5 W^7 + x_6 W^2 + x_7 W^5 \quad \text{or}$$

$$X_3 = (x_0 + x_2 W^6 + x_4 W^4 + x_6 W^2) + W^3(x_1 + x_3 W^6 + x_5 W^4 + x_7 W^2) = Y_3 + W^3 Z_3.$$

$$\text{Also } X_4 = x_0 + x_1 W^4 + x_2 + x_3 W^4 + x_4 + x_5 W^4 + x_6 + x_7 W^4. \quad \text{or}$$

$$X_4 = (x_0 + x_2 + x_4 + x_6) + W^4(x_1 + x_3 + x_5 + x_7) = Y_0 + W^4 Z_0.$$

It is seen that X_4 which is the spectral value at the Nyquist frequency is given in terms of the zeroth spectral values of the shorter sequence. In this way we use the 4 spectral values of the shorter sequences to give the 8 spectral values of the original sequence.

$$X_5 = x_0 + x_1 W^5 + x_2 W^2 + x_3 W^7 + x_4 W^4 + x_5 W^9 + x_6 W^6 + x_7 W^{11}$$

$$X_5 = (x_0 + x_2 W^2 + x_4 W^4 + x_6 W^6) + W^5(x_1 + x_3 W^2 + x_5 W^4 + x_7 W^6) = Y_1 + W^5 Z_1$$

$$X_6 = x_0 + x_1 W^6 + x_2 W^4 + x_3 W^{10} + x_4 + x_5 W^6 + x_6 W^4 + x_7 W^{10}$$

$$X_6 = (x_0 + x_2 W^4 + x_4 + x_6 W^4) + W^6(x_1 + x_3 W^4 + x_5 + x_7 W^4) = Y_2 + W^6 Z_2$$

$$X_7 = x_0 + x_1 W^7 + x_2 W^6 + x_3 W^{13} + x_4 W^4 + x_5 W^{11} + x_6 W^2 + x_7 W^9$$

$$X_7 = (x_0 + x_2 W^6 + x_4 W^4 + x_6 W^2) + W^7(x_1 + x_3 W^6 + x_5 W^4 + x_7 W^2) = Y_3 + W^7 Z_3.$$

To restate the finite Fourier transform of the long sequence in terms of the two shorter ones is given by

$$X_0 = Y_0 + Z_0 \quad X_4 = Y_0 + W^4 Z_0$$

$$X_1 = Y_1 + W Z_1 \quad X_5 = Y_1 + W^5 Z_1$$

$$X_2 = Y_2 + W^2 Z_2 \quad X_6 = Y_2 + W^6 Z_2$$

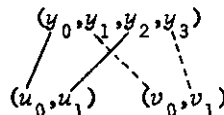
$$X_3 = Y_3 + W^3 Z_3 \quad X_7 = Y_3 + W^7 Z_3$$

Equations (8)

The above technique alone can be used to reduce the number of operations required to calculate $(X_0, X_1, X_2, X_3, X_4, X_5, X_6, X_7)$. This follows because we are doing two transforms each of length 4 instead of one transform of length 8. The time using this technique is proportional to $2(4)^2 = 32$ and not $(8)^2 = 64$.

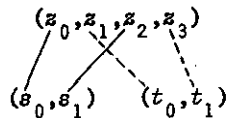
The method of FFT is to carry this decimating procedure to the point of numerous sequences each of length 2. For the above case of $N = 8$, we note that the sequences (y_0, y_1, y_2, y_3) and (z_0, z_1, z_2, z_3) can both be divided or halved into 4 sequences of length 2. The equations which give the spectral values (Y_0, Y_1, Y_2, Y_3) and (Z_0, Z_1, Z_2, Z_3) in terms of the spectral values of the shorter sequences will now be developed.

The sequence $(y_0, y_1, y_2, y_3) = (x_0, x_2, x_4, x_6)$ is halved as follows:



We note that $u_0 = y_0$, $u_1 = y_2$ and $v_0 = y_1$, $v_1 = y_3$. It follows that $u_0 = x_0$, $u_1 = x_4$, $v_0 = x_2$ and $v_1 = x_6$.

For the other sequence $(z_0, z_1, z_2, z_3) = (x_1, x_3, x_5, x_7)$ the procedure is the same.



As indicated above $s_0 = z_0$, $s_1 = z_2$, $t_0 = z_1$, and $t_1 = z_3$. It follows that $s_0 = x_1$, $s_1 = x_5$, $t_0 = x_3$ and $t_1 = x_7$.

The spectral values or finite Fourier transform of the 4 sequences (u_0, u_1) , (v_0, v_1) , (s_0, s_1) and (t_0, t_1) are very easily obtained as follows:

$$U_0 = u_0 + u_1$$

$$U_1 = u_0 - u_1 = u_0 + W^4 u_1$$

It follows that we have

$$U_0 = u_0 + u_1 = x_0 + x_4$$

$$U_1 = u_0 + W^4 u_1 = x_0 + W^4 x_4$$

Equations (9)

$$V_0 = v_0 + v_1 = x_2 + x_6$$

$$V_1 = v_0 + W^4 v_1 = x_2 + W^4 x_6$$

$$S_0 = s_0 + s_1 = x_1 + x_5$$

$$S_1 = s_0 + W^4 s_1 = x_1 + W^4 x_5$$

$$T_0 = t_0 + t_1 = x_3 + x_7$$

$$T_1 = t_0 + W^4 t_1 = x_3 + W^4 x_7.$$

Equations (9)

We now develop the equations giving (Y_0, Y_1, Y_2, Y_3) in terms of (U_0, U_1) and (V_0, V_1) . These are followed by the equations giving (Z_0, Z_1, Z_2, Z_3) in terms of (S_0, S_1) and (T_0, T_1) .

$$Y_0 = y_0 + y_1 + y_2 + y_3 = x_0 + x_2 + x_4 + x_6 = U_0 + V_0$$

$$Y_1 = y_0 + y_1 W^2 + y_2 W^4 + y_3 W^6 = x_0 + x_2 W^2 + x_4 W^4 + x_6 W^6 = U_1 + W^2 V_1$$

$$Y_2 = y_0 + y_1 W^4 + y_2 + y_3 W^4 = x_0 + x_2 W^4 + x_4 + x_6 W^4 = U_0 + W^4 V_0$$

$$Y_3 = y_0 + y_1 W^6 + y_2 W^4 + y_3 W^{10} = x_0 + x_2 W^6 + x_4 W^4 + x_6 W^{10} = U_1 + W^6 V_1$$

Equations (10)

$$Z_0 = z_0 + z_1 + z_2 + z_3 = x_1 + x_5 + x_3 + x_7 = S_0 + T_0$$

$$Z_1 = z_0 + z_1 W^2 + z_2 W^4 + z_3 W^6 = x_1 + x_5 W^2 + x_3 W^4 + x_7 W^6 = S_1 + W^2 T_1$$

$$Z_2 = z_0 + z_1 W^4 + z_2 + z_3 W^4 = x_1 + x_5 W^4 + x_3 + x_7 W^4 = S_0 + W^4 T_0$$

$$Z_3 = z_0 + z_1 W^6 + z_2 W^4 + z_3 W^{10} = x_1 + x_5 W^6 + x_3 W^4 + x_7 W^{10} = S_1 + W^6 T_1$$

Although the equation for Y_3 may appear different from the previous equation for Y_3 it is not, for $W^{10} = W^8 W^2 = W^2$ as $W^8 = 1$.

The basis of the Fast Fourier Transform or FFT method for $N = 8$ is to use the sets of equations (9), (10) and (8) to calculate the finite Fourier transform of $(x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ and not equation (7). That is, the first step is to calculate $(U_0, S_0, V_0, T_0, U_1, S_1, V_1, T_1)$ using equations (9). These 8 spectral values are stored in the positions (complex)

occupied by the original data values for they are no longer needed. The second step is to calculate $(Y_0, Z_0, Y_2, Z_2, Y_4, Z_4, Y_6, Z_6)$ from $(U_0, S_0, V_0, T_0, U_1, S_1, V_1, T_1)$ using equations (10). The third step computes the required finite Fourier transform $(X_0, X_4, X_2, X_6, X_1, X_5, X_3, X_7)$ from $(Y_0, Z_0, Y_2, Z_2, Y_4, Z_4, Y_6, Z_6)$ using equations (8). Using an additional step of rearranging to give $(X_0, X_1, X_2, X_3, X_4, X_5, X_6, X_7)$ the transform is completed. From equations (9), (10) and (8) we see that each step requires 8 multiply-add operations and there are 3 such steps. It follows that the finite Fourier transform is obtained in $8 \times 3 = 24$ multiply-add operations instead of 64 multiply-add operations if the straight forward approach were used. The above technique widely known as the FFT can be generalized to handle sequences of any length N (where $N = 2^n$) and the number of operations is only Nn and not N^2 .

Examples of computer programs using the FFT method are NLOGN described in *Multichannel Time Series Analysis* by E. A. Robinson and COOL described in *Finite Fourier Transform Theory* by D. W. McCowan.

The basic properties of the finite Fourier transform and the very quick FFT procedure to calculate it should now be clearly understood. With this in mind we now give some examples of its application in geophysical processing. Of the many possible examples of application we choose to discuss only two, namely, VIBROSEIS® correlation and very briefly frequency domain deconvolution.

VIBROSEIS® CORRELATION USING THE FFT

In conventional Vibroseis processing the sweep s_i is cross-correlated with recorded data x_i to produce the correlated seismic trace $C'_{ss}(n)$. With this example, we hope to remove some of the vagueness associated with the term "processing in the frequency domain". It will be noted that, for this case, the resulting correlated trace will be the same whether the straight correlation (multiply-add) or the finite Fourier transform is used. We now take up a specific example and will compare the timing of the two approaches.

Let us suppose the sweep is 7 seconds long and that the data is digitally recorded to 14 seconds at a 4 ms. sampling interval. It follows that the data is 3500 samples long and the sweep is 1750 samples long. We represent these two signals as follows:

data $x_1, x_2, x_3, \dots, x_{1750}, \dots, x_{3500}$

sweep $s_1, s_2, s_3, \dots, s_{1750}$.

For simplicity of notation we use x_1 and s_1 , instead of x_0 and s_0 , to denote the first values.

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A common practice, for the above example, is to calculate the correlated trace only for the first 1751 values. In the time domain this is accomplished by shifting - multiplying - adding until the s_{1750} value is shifted under the x_{3500} value. That is,

$$C'_{xs}(n) = \sum_{t=1}^{1750} x_{t+n} s_t \quad \text{for } n = 0, 1, 2, \dots, 1750.$$

is calculated. To perform this cross-correlation on one 14 second data trace would require 3,064,250 multiply-add operations. This follows since each correlated output value requires 1750 multiply - adds and therefore 1751 output values require $(1751) \times (1750) = 3,064,250$ multiply-adds. It follows that two data traces correlated with the same sweep would require $2 \times 3,064,250 = 6,128,500$ multiply-adds. If the time required for one multiply-add cycle is 1.0×10^{-6} seconds, then the time to obtain the two correlated traces is 6.13 seconds. It should be noted that the time for one cycle will be hardware dependent.

We can also obtain these two correlated output traces via the finite Fourier transform. Using the FFT algorithm it is quite possible that the frequency domain approach is faster. If it were, then from the standpoint of economics (which is the only valid criterion as the results are the same) the finite Fourier transform method should be used. The time using the finite Fourier transform would also be hardware dependent. As an example, for the above 7 second sweep and 14 second data the time is only .45 seconds using the IBM 360/44 and 2938 Array Processor.

The above time of .45 seconds was used up performing the following operations:

1. Take the two 14 second data traces, namely,

$x_1, x_2, x_3, \dots, x_{3500}$ and $y_1, y_2, y_3, \dots, y_{3500}$ and make one complex list $(z_1, z_2, z_3, \dots, z_{3500})$, where $z_t = x_t + iy_t$ for $t = 1, 2, 3, \dots, 3500$.

As 3500 is not a power of 2 we pad the sequence z_t out to $4096 = 2^{12}$, (add 596 zeros). The sweep is also padded to 4096 (add $1750 + 596 = 2346$ zeros). It should be noted that the zeros added to z_t are only to obtain the appropriate number of samples for which the FFT algorithm is most efficient. They do not serve the same purpose as the zeros mentioned in property 6. The procedure is a little different here as we do not require the complete cross - correlation. That is, we only require

$$C'_{zs}(n) = \sum_{t=1}^{4096} z_{t+n} s_t \quad \text{for } n = 0, 1, 2, \dots, 1750.$$

We must be aware of the inherent periodicity of the frequency domain approach, that is, from property 6 we know that the inverse transform of the product $Z_k S_k^*$ is the circular correlation of the zero filled sequences. It follows that with the minimum amount of zeros we have added that many of the values of the full inverse of $Z_k S_k^*$ for $k = 1, 2, \dots, 4096$, namely $C'_{zs}(n)$ for $n > 1750$ are distorted by the wrap around effect. But since

the final correlated trace length is to be only the first 1751 values it follows that the zeroes added is sufficient.

2. We now obtain the finite Fourier transform of $(z_1, z_2, z_3, \dots, z_{3500}, 0, 0, 0, \dots, 0_N)$ $N = 4096 = 2^{12}$.

We use the FFT method and the result is Z_k for $k = 1, 2, 3, \dots, 4096$. We consider the case of using the same sweep s_t for all traces and hence its finite Fourier transform S_k for $k = 1, 2, 3, \dots, 4096$ is considered to be in storage. Hence in this step it is only the one forward transform ($N = 4096$) of the complex sequence z_t which contributes to the total time of .45 seconds. It should be recalled that .45 seconds is the time required to output two correlated traces $C'_{zs}(n)$ and $C'_{ys}(n)$, each 1751 samples long.

3. We now perform the complex multiplication $Z_k S_k^*$ for $k = 1, 2, 3, \dots, 4096$. These calculations also contribute to the total time of .45 seconds.
4. The inverse transform of the product $Z_k S_k^*$ for $k = 1, 2, 3, \dots, 4096$ is calculated. By property 6 this gives the circular correlation

$$C'_{zs}(n) = \sum_{t=1}^{4096} z_{t+n} s_t = \sum_{t=1}^{4096} x_{t+n} s_t + i \sum_{t=1}^{4096} y_{t+n} s_t.$$

It should be noted that property 6 also holds for z_t a complex sequence and S_t a real sequence. That is $Z_k S_k^*$ is the frequency domain representation of $C'_{zs}(n) = C'_{zs}(n) + i C'_{ys}(n)$. If both sequences are complex, then it can be shown that $Z_k S_{N-k}$ and not $Z_k S_k^*$ is the frequency domain representation of $C'_{zs}(n)$. In other words, the general frequency representation is $Z_k S_{N-k}$ but for S_t real we have by property 5 that $S_{N-k} = S_k^*$.

It follows from the above equation that the real part of the inverse is the one correlated trace $C'_{zs}(n)$ and the imaginary part of the inverse is the other correlated trace $C'_{ys}(n)$. In step 4 it is the one inverse transform that contributes to the total time of .45 seconds.

In this example the correlation as carried out by the finite Fourier transform and FFT is faster, .45 seconds compared to 6.13 seconds for straight correlation. Of course, the timing will be very much dependent on the users hardware. The transform method does provide one with the opportunity to do a straight forward frequency domain deconvolution (zero phase as required by Vibroseis) before the inverse transform is calculated. We shall now briefly discuss frequency domain deconvolution.

FREQUENCY DOMAIN DECONVOLUTION

The basic assumption of least squares deconvolution is that the earth's frequency spectrum is white and hence the colouring (non-white character) of the frequency spectrum of the returned seismic trace is due only to the seismic wavelet. This may not be the true situation for all geologic

sections, that is, the earth spectrum may be quite coloured and these variations should be retained as they will contain the desired geologic information.

Although the separation of the returned spectrum into its earth component and wavelet component is very difficult, an approximate solution is the frequency domain deconvolution process. In this process the finite Fourier transform of the seismic trace x_t , say $t = 1, 2, 3, \dots, 1024$ is calculated giving the spectral values X_k for $k = 1, 2, 3, \dots, 1024$. Now X_k is a complex number and it can be written in polar form to display its "length" $|X_k|$ and its "angle" θ_k as follows:

$$X_k = |X_k| e^{i\theta_k} \quad \text{for } k = 1, 2, 3, \dots, 1024.$$

The "length of X_k ", namely, $|X_k|$ is the amplitude spectrum and the "angle of X_k ", namely θ_k is the phase spectrum of x_t . The trace x_t is the convolution of the earth's reflectivity sequence e_t with the seismic wavelet w_t . From property 7 we have that the frequency domain representation of this convolution is related to its respective components as follows:

$$X_k = E_k W_k.$$

Writing each complex number in its polar form we have

$$|X_k| e^{i\theta_k} = |E_k| e^{i\epsilon_k} |W_k| e^{i\omega_k}$$

or

$$|X_k| e^{i\theta_k} = |E_k| |W_k| e^{i(\epsilon_k + \omega_k)}.$$

It follows that the amplitude spectrum of the convolutional sequence is the product of component amplitude spectra, while the phase spectrum of the convolutional sequence is the sum of the component phase spectra. That is,

$$|X_k| = |E_k| |W_k|$$

and

$$\theta_k = \epsilon_k + \omega_k.$$

The amplitude spectrum might appear as shown in Figure 1.

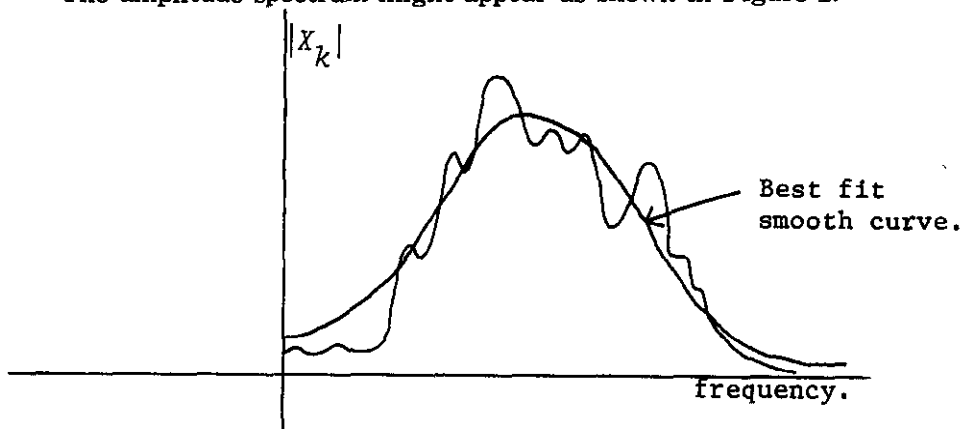


FIG. 1.—Raw amplitude spectrum and best fit smooth curve.

The technique used by frequency domain deconvolution (to remove only the colouring due to the wavelet which should have a fairly smooth spectra, assuming no reverberations) is to fit a smooth curve $|S_k|$ to the amplitude spectrum, as shown in Fig. 1. The next step is to take the inverse of this curve, namely, calculate $1/|S_k|$ and weight the original spectrum $|X_k|$ with it. The result of this is the new spectrum $|X_k| / |S_k|$ which might appear, as shown in Fig. 2.

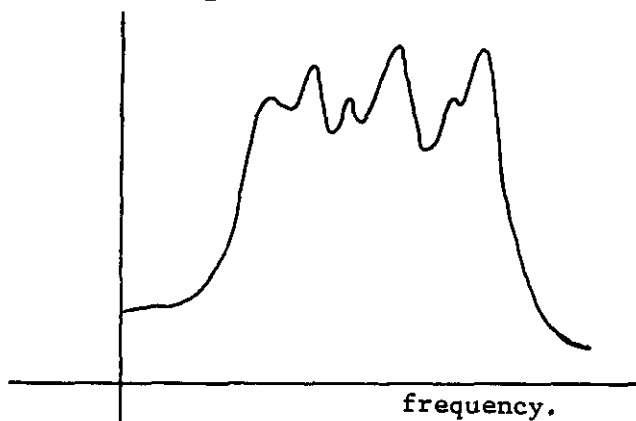


FIG. 2.—Deconvolved amplitude spectra.

As can be seen in Fig. 2, the resultant spectrum is not completely white (flat). Hopefully, the remaining variation is due to the coloured earth spectrum. If this variation is due to the geologic section, then it follows that the seismic trace obtained by inverting the deconvolved spectrum should contain the desired geologic information. If on the other hand the

spectrum was considerably whitened (by least squares deconvolution with very little white noise added), then it is possible that some of the desired elements in the earth's spectrum might be attenuated.

This method of deconvolution is very suitable for Vibroseis data as zero phase (no phase shifting) deconvolution can be performed quite easily.

CONCLUSION

There are numerous other applications of the finite Fourier transform to geophysical data processing. For example, the two dimensional transform of the seismic section to attenuate or enhance events with different velocities. It should be noted that once the procedure for one dimensional transform is available there is nothing new with two dimensional transforms. Another example, not so well established, is CDP stacking in the frequency domain. Instead of straight vector addition of $x_k = |x_k| e^{i\theta_k}$ there are numerous other possible ways to average. Since the amplitude and phase spectra are separate they can be averaged independently. One such averaging technique would be to take the geometric mean of the amplitude spectra and the arithmetic mean of the phase spectra. We are sure the reader can think of other averaging methods, which could be tested to see if they give better results.

All in all it is quite likely the use of the finite Fourier transform will continue to grow. It is hoped that this article has helped the reader to understand the mechanics of doing the transform. It is also hoped that the description of some of its basic properties will help to illuminate some of the more sophisticated frequency domain processes.

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